SEP-optimal Adaptive Gain and Transmit Power Amplify-and-Forward Relaying

(Invited Paper)

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***Abstract*—Amplify-and-forward (AF) relay based cooperation has been investigated in the literature given its simplicity and practicality. Two models for AF, namely, fixed gain and fixed power relaying, have been extensively studied. In fixed gain relaying, the relay gain is fixed but its transmit power varies as a function of the source-relay (SR) channel gain. In fixed power relaying, the relay's instantaneous transmit power is fixed, but its gain varies. We propose a general AF cooperation model in which an average transmit power constrained relay jointly adapts its gain and transmit power as a function of the channel gains. We derive the optimal AF gain policy that minimizes the fading-averaged symbol error probability (SEP) of MPSK and present insightful and tractable lower and upper bounds for it. We then analyze the SEP of the optimal policy. Our results show that the optimal scheme is up to 39.7% and 47.5% more energy-efficient than fixed power relaying and fixed gain relaying, respectively. Further, the weaker the direct source-destination link, the greater are the energy-efficiency gains.**

I. Introduction

Relay-based cooperative communications exploits spatial diversity to combat wireless fading. In amplify-and-forward (AF) relaying, the relay amplifies the noisy and faded signal it receives from the source and transmits it to the destination; it does not decode the source's message [1]–[8]. AF is considered to be simple and, yet, achieves full diversity.

Two AF relaying models have been extensively investigated in the literature, namely, fixed power and fixed gain relaying. In fixed power relaying, the relay adjusts its gain as a function of the source-relay (SR) channel gain such that its instantaneous transmit power, when averaged over the noise in the SR channel and the data symbols, is fixed [1]–[6]. On the other hand, in fixed gain relaying, the relay gain is fixed. As a result, its instantaneous transmit power depends on the SR channel gain. The fixed gain is set so that the relay meets an average power constraint [7], [8]. While fixed gain relaying is considered easier to implement, fixed power relaying simplifies the design of a power amplifier at the relay.

In this paper, we generalize the AF relaying model and consider an AF relay that adapts its gain as well as power as a function of the channel gains, subject to an average power constraint. We refer to this as adaptive relay gain and transmit power (ARGTP) relaying. While the average power constraint model has been used in several wireless systems [9], its role in AF relaying has not been fully explored to the best of the our knowledge. For example, in [10] and the references therein, the total power of the source and the AF relay is constrained, but a relay is still subject to a fixed instantaneous power constraint.

While the model considered in [11] also considers an average power constraint and varies the relay gain as a function of the SR and relay-destination (RD) channel gains, our design objectives and the depth of our results are vastly different. In [11], the relay adjusts its transmit power as a function of the SR and RD channel gains so as to maximize the end-to-end SNR. We instead focus on minimizing the fading-averaged end-to-end SEP. In [11], the optimal gain was determined numerically as a function of the channel gains using computationally efficient quasi-convex optimization techniques. Instead, we analytically characterize the SEP-optimal relay gain and also develop insightful closed-form upper and lower bounds for the relay gain. Another important contribution of the paper is the SEP analysis of the optimal policy. Our results show that ARGTP leads to significant energy savings over fixed power relaying and fixed gain relaying. The savings increase when the direct source-destination (SD) link is absent.

The paper is organized as follows. Section II describes our system model. Section III derives the optimal ARGTP relaying rule and its SEP. Our results and conclusions follow in Sections IV and V. Mathematical details are relegated to the Appendix.

II. System Model

Consider a three node system consisting of a source S, a destination D, and a half-duplex relay R. Each node is equipped with a single transmit or receive antenna. The SR, RD, and SD channels are assumed to be block fading channels that undergo independent frequency-flat Rayleigh fading. They need not be statistically identical. All transmissions occur over the same bandwidth. The relay is assumed to know its local SR and RD channel gains [11], [12]. In practice, these can be acquired by a training protocol; see [13] and references therein. However, the relay need not know the SD channel gain.

We use the following notation henceforth. The probability of an event A is denoted by $\Pr(A)$. For a random variable (RV) X, its probability density function (PDF), expectation, and variance are denoted by $p_X(x)$, $\mathbf{E}[X]$, and $\mathbf{var}[X]$, respectively. $\mathcal{CN}(0, \sigma^2)$ represents a zero-mean circular symmetric complex Gaussian RV with variance σ^2, and x^* denotes

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complex conjugate of x.

A. Cooperative AF Protocol

The cooperative AF relaying protocol occurs over two phases, and is similar to that for fixed gain or fixed power AF relaying [1]–[6]. In the first phase, the source broadcasts a data symbol α that is drawn with equal probability from the M-ary PSK (MPSK) constellation of size M. The received signals y_{sd} and y_{sr} at the destination and relay, respectively, are given by

$$y_{sd} = \sqrt{P_s} h_{sd} \alpha + n_{sd}, \tag{1}$$

$$y_{sr} = \sqrt{P_s} h_{sr} \alpha + n_{sr}, \tag{2}$$

where P_s is the source transmit power, h_{sd} is the SD channel gain and is a $\mathcal{CN}(0, \sigma_{sd}^2)$ RV, h_{sr} is the SR channel gain and is a $\mathcal{CN}(0, \sigma_{sr}^2)$ RV, and $|\alpha|^2 = 1$. The additive noise terms n_{sr} and n_{sd} are $\mathcal{CN}(0, 1)$ RVs, and are independent of each other and the channel gains.

In the second phase, the relay amplifies the signal it receives, y_{sr}, by a factor $\sqrt{\beta \overline{P}_r}$, where $\overline{P}_r$ is the average relay transmit power. Therefore, the destination receives the signal y_{rd}, which is given by

$$\begin{aligned} y_{rd} &= \sqrt{\beta \overline{P}_r} h_{rd} y_{sr} + n_{rd}, \\ &= \sqrt{\beta \overline{P}_r P_s} h_{sr} h_{rd} \alpha + \sqrt{\beta \overline{P}_r} h_{rd} n_{sr} + n_{rd}. \end{aligned} \tag{3}$$

Let $\gamma_{sd} \triangleq |h_{sd}|^2 P_s$, $\gamma_{sr} \triangleq |h_{sr}|^2 P_s$, and $\gamma_{rd} \triangleq \overline{P}_r |h_{rd}|^2$. Further, let $\overline{\gamma}_{sd} \triangleq \mathbf{E}[\gamma_{sd}]$, $\overline{\gamma}_{sr} \triangleq \mathbf{E}[\gamma_{sr}]$, and $\overline{\gamma}_{rd} \triangleq \mathbf{E}[\gamma_{rd}]$.

B. General AF Relaying Model

In fixed power AF relaying [1], [2], [5], the relay gain for the above relay transmission model is given by $\beta = \frac{1}{\gamma_{sr}+1}$. On the other hand, in fixed gain AF relaying, the relay gain equals $\beta = \frac{1}{\overline{\gamma}_{sr}+1}$.

In our model, however, the relay gain is a function of the local SR and RD channel gains, γ_{sr} and γ_{rd}. As mentioned, we do not assume that the relay knows the SD channel gain, as this would require channel state feedback from the destination. Henceforth, the relay gain will be denoted by $\beta(\gamma_{sr}, \gamma_{rd})$ in order to explicitly show its dependence on γ_{sr} and γ_{rd}. The relay is subject to an average relay transmit power constraint, which is $\mathbf{E}[\beta(\gamma_{sr}, \gamma_{rd})(\gamma_{sr} + 1)] = 1$.

The destination coherently determines α using its two observables y_{sd} and y_{rd}. We assume that it knows the SR, RD, and SD channel gains [1]–[3]. The SNR γ_E at the destination receiver when it employs maximal ratio combining is given by

$$\gamma_E = \gamma_{sd} + \frac{\gamma_{sr}\gamma_{rd}}{\gamma_{rd} + \beta(\gamma_{sr}, \gamma_{rd})^{-1}}. \tag{4}$$

III. Optimal ARGTP Relaying and SEP Analysis

We first derive the SEP-optimal relay gain policy. The fading-averaged SEP for MPSK at the destination is given by [14, (8.23)]

$$\text{SEP} = \frac{1}{\pi} \int_0^{\left(\frac{M-1}{M}\right)\pi} \mathbf{E}\left[\exp\left(-\gamma_E \frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\theta}\right)\right] d\theta. \tag{5}$$

Averaging over γ_{sd}, which is an exponential RV that is independent of γ_{sr}, γ_{rd}, and $\beta(\gamma_{sr}, \gamma_{rd})$, we get

$$\text{SEP} = \frac{1}{\pi} \int_0^{\left(\frac{M-1}{M}\right)\pi} \frac{\mathbf{E}\left[\exp\left(-\frac{\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta(\gamma_{sr},\gamma_{rd})^{-1}} \frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\theta}\right)\right]}{1 + \overline{\gamma}_{sd}\frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\theta}} d\theta. \tag{6}$$

The SEP expression in (6) cannot be simplified further because the relay gain is itself a function of γ_{sr} and γ_{rd}, and it is this function that we seek to optimize. To gain further insights, we derive below an analytically tractable upper bound for the SEP and shall minimize it instead. Using the inequality $\sin^2\theta \le 1$ only for the term inside the expectation in the integrand in (6), we get

$$\text{SEP} \le \text{SEP}_0 \mathbf{E}\left[\exp\left(-\frac{\gamma_{sr}\gamma_{rd}}{\gamma_{rd} + \beta(\gamma_{sr}, \gamma_{rd})^{-1}} \sin^2\left(\frac{\pi}{M}\right)\right)\right]. \tag{7}$$

Here, it can be shown that [6, (9)]

$$\begin{aligned} \text{SEP}_0 &= \frac{1}{\pi} \int_0^{\left(\frac{M-1}{M}\right)\pi} \left(1 + \overline{\gamma}_{sd} \frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\theta}\right)^{-1} d\theta = \frac{M-1}{M} \\ &- \frac{1}{\sqrt{1 + \frac{\csc^2\left(\frac{\pi}{M}\right)}{\overline{\gamma}_{sd}}}} \left(\frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{\cot\left(\frac{\pi}{M}\right)}{\sqrt{1 + \frac{\csc^2\left(\frac{\pi}{M}\right)}{\overline{\gamma}_{sd}}}}\right)\right). \end{aligned} \tag{8}$$

A. AF Relaying Optimization

Since SEP_0 does not depend on the relay gain, the SEP minimization problem reduces to finding the optimal function $\beta_{\text{opt}} : (\mathbb{R}^+)^2 \to \mathbb{R}$, which is a function of two variables γ_{sr} and γ_{rd}, that minimizes the expectation term in (7). Mathematically, the optimization problem can be stated as

$$\min_{\beta} \mathbf{E}\left[\exp\left(-\frac{\gamma_{sr}\gamma_{rd}}{\gamma_{rd} + \beta(\gamma_{sr}, \gamma_{rd})^{-1}} \sin^2\left(\frac{\pi}{M}\right)\right)\right] \tag{9}$$

$$\text{such that } \mathbf{E}[\beta(\gamma_{sr}, \gamma_{rd})(\gamma_{sr} + 1)] = 1, \text{ and} \tag{10}$$

$$\beta(\gamma_{sr}, \gamma_{rd}) \ge 0, \text{ for all } \gamma_{sr} \ge 0, \gamma_{rd} \ge 0. \tag{11}$$

The optimal solution is as follows.

Result *1:* Let

$$\phi(x) \triangleq \exp\left(\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\gamma_{rd} + x^{-1}}\right)(x\gamma_{rd} + 1)^2 - \frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\lambda(\gamma_{sr} + 1)}. \tag{12}$$

For $\gamma_{rd} \ge \mathcal{B}(\gamma_{sr})$, $\phi(x)$ has a unique positive root $x_0(\gamma_{sr}, \gamma_{rd})$, and

$$\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd}) = \begin{cases} x_0(\gamma_{sr}, \gamma_{rd}), & \gamma_{rd} \ge \mathcal{B}(\gamma_{sr}) \\ 0, & \text{otherwise} \end{cases}, \tag{13}$$

where

$$\mathcal{B}(\gamma_{sr}) \triangleq \frac{\lambda}{\sin^2\left(\frac{\pi}{M}\right)}\left(1 + \frac{1}{\gamma_{sr}}\right). \tag{14}$$

The Lagrange multiplier λ is chosen to satisfy the average power constraint in (10).

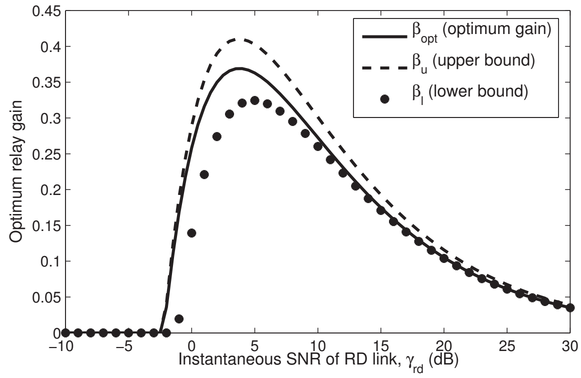

Fig. 1. Comparison of optimum relay gain β_{opt}, its upper bound β_u, and its lower bound β_l, as a function of the instantaneous RD link SNR γ_{rd} ($\gamma_{sr} = 0.5$ dB, $\lambda = 0.1$, and QPSK).

Proof: The proof is relegated to Appendix A. ■

Note that the Lagrange multiplier depends on the channel fading statistics of the SR and RD links and not their instantaneous channel gains. It, therefore, needs to be numerically computed only once. However, $x_0(\gamma_{sr}, \gamma_{rd})$ needs to be computed by numerically solving (12) for each realization of γ_{sr} and γ_{rd}. We now present upper and lower bounds for $\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd})$ that have an explicit closed-form characterization. These provide new insights into the structure of the optimal AF relaying scheme.

Result *2:* The optimal AF relay gain β_{opt} is upper and lower bounded as follows:

$$\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd}) \leq \beta_u(\gamma_{sr}, \gamma_{rd}) = \begin{cases} \dfrac{-1+\sqrt{1+\left[\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\lambda(\gamma_{sr}+1)}-1\right]\left(1+\gamma_{sr}^2\sin^4\left(\frac{\pi}{M}\right)\right)}}{\gamma_{rd}\left(1+\gamma_{sr}^2\sin^4\left(\frac{\pi}{M}\right)\right)}, & \gamma_{rd} \geq \mathcal{B}(\gamma_{sr}), \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

$$\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd}) \geq \beta_l(\gamma_{sr}, \gamma_{rd}) = \begin{cases} \dfrac{\sqrt{\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\lambda(\gamma_{sr}+1)}}\exp\left(-\frac{\gamma_{sr}\sin^2\left(\frac{\pi}{M}\right)}{2}\right)-1}{\gamma_{rd}}, & \gamma_{rd} \geq \mathcal{B}_1(\gamma_{sr}), \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

where $\mathcal{B}_1(\gamma_{sr}) = \mathcal{B}(\gamma_{sr})\exp\left(\gamma_{sr}\sin^2\left(\frac{\pi}{M}\right)\right)$.

Proof: The proof is given in Appendix B. ■

The above two bounds and β_{opt} are shown in Figure 1. We see that the bounds track β_{opt} well. They together show that $\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd})$ decays as $\gamma_{rd}^{-\frac{1}{2}}$ for large γ_{rd}. Their closed-form characterization also makes them amenable to a practical implementation.

B. SEP Analysis of Optimal ARGTP Relaying

Expanding the expectation term in the integrand in (6) results in a triple-integral expression for the SEP. Using the inequality $\sin^2\theta \leq 1$, we get the following SEP upper bound SEP_u:

$$\text{SEP}_u = \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}}\int_0^\infty\int_0^\infty \exp\left(-\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\gamma_{rd}+\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})^{-1}}\right) \times \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)\exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right)d\gamma_{sr}\,d\gamma_{rd}. \quad (17)$$

It is in the form of a double-integral, which can be simplified as shown below.

Result *3:* The SEP of optimal ARGTP relaying is upper bounded as follows:

$$\text{SEP} \leq \text{SEP}_u \leq \text{SEP}_{uu} \triangleq I_1 + \frac{\text{SEP}_0}{\overline{\gamma}_{rd}\overline{\gamma}_{sr}}\int_0^\infty\left[b_1^2\text{Ei}\left(\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right) + b_2\overline{\gamma}_{rd}\exp\left(-\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right) + 2b_1\sqrt{b_2\pi\overline{\gamma}_{rd}}\,\text{erfc}\left(\sqrt{\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}}\right)\right]\exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)d\gamma_{sr}, \quad (18)$$

where SEP_0 is given by (8), $b_1 = \frac{\gamma_{sr}^2\sin^4\left(\frac{\pi}{M}\right)}{1+\gamma_{sr}^2\sin^4\left(\frac{\pi}{M}\right)}$, $b_2 = \frac{\gamma_{sr}\sin^2\left(\frac{\pi}{M}\right)}{\lambda(\gamma_{sr}+1)\left(1+\gamma_{sr}^2\sin^4\left(\frac{\pi}{M}\right)\right)}$, and

$$I_1 = \text{SEP}_0\left[1-\frac{2}{\overline{\gamma}_{sr}}\exp\left(-\frac{\lambda}{\overline{\gamma}_{rd}\sin^2\left(\frac{\pi}{M}\right)}\right)\sqrt{\frac{\lambda\overline{\gamma}_{sr}}{\overline{\gamma}_{rd}\sin^2\left(\frac{\pi}{M}\right)}} \times K_1\left(\sqrt{\frac{4\lambda}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}\sin^2\left(\frac{\pi}{M}\right)}}\right)\right]. \quad (19)$$

Here, $\text{erfc}(\cdot)$ is the complementary error function, $\text{Ei}(\cdot)$ is the exponential integral [15, (8.211.1)], and $K_1(\cdot)$ denotes the modified Bessel function of second kind and first order [16, (9.6)].

Proof: The derivation is given in Appendix C. ■

IV. Numerical Results and Discussion

We now verify our analytical results using Monte Carlo simulations that use up to 10^6 fading and noise realizations, and benchmark the performance of ARGTP relaying with both fixed power and fixed gain relaying. In the simulations, we set $\mathbf{E}\left[|h_{sr}|^2\right] = 1$ and $\mathbf{E}\left[|h_{rd}|^2\right] = 1$. Recall that the additive noise power is normalized to unity. We consider below the following two scenarios: (i) where the SD link is comparable in strength to the SR and RD links ($\mathbf{E}\left[|h_{sd}|^2\right] = 1$), and (ii) where the SD link is absent ($\mathbf{E}\left[|h_{sd}|^2\right] = 0$).

A. With SD link

Figure 2 plots the SEP of ARGTP relaying as a function of the average relay transmit power for QPSK. The results from Monte Carlo simulations and the SEP upper bounds SEP_u in (7) and SEP_{uu} are also plotted. We see that SEP_u and SEP_{uu} are within 0.3 dB and 0.7 dB, respectively, at an SEP of 10^{-2}.

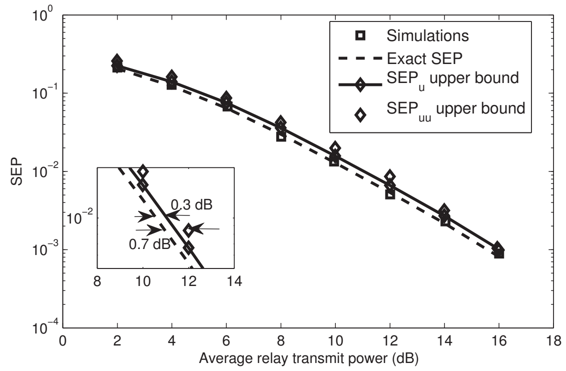

Fig. 2. SEP and its bounds as a function of average relay transmit power, $\overline{P}_r$ ($\overline{\gamma}_{sr} = \overline{\gamma}_{rd} = \overline{\gamma}_{sd} = P_s$, $P_s = \overline{P}_r$, and QPSK).

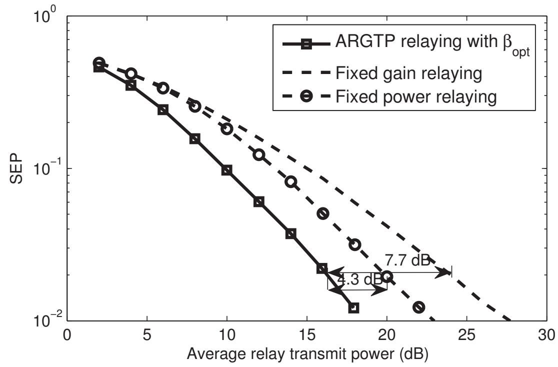

Fig. 4. Without SD link: Comparison of SEPs of AF relaying schemes as a function of average relay transmit power, $\overline{P}_r$ ($\overline{\gamma}_{sr} = \overline{\gamma}_{rd} = P_s$, $\overline{\gamma}_{sd} = 0$, and QPSK).

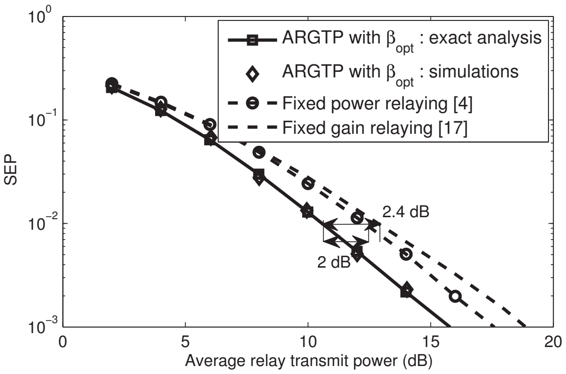

Fig. 3. With SD link: Comparison of SEPs of AF relaying schemes as a function of average relay transmit power, $\overline{P}_r$ ($P_s = \overline{P}_r$ and $\overline{\gamma}_{sr} = \overline{\gamma}_{rd} = \overline{\gamma}_{sd} = P_s$, and QPSK).

Figure 3 plots the SEP as a function of the average relay transmit power for QPSK. It compares the SEPs of ARGTP relaying, fixed gain relaying, and fixed power relaying when $P_s = \overline{P}_r$. We see that ARGTP relaying requires 2.0 dB (36.9%) less power than fixed power relaying and 2.4 dB (42.5%) less power than fixed gain relaying at an SEP of 10^{-2}. For 8PSK (figure not shown), the power savings increase to 2.2 dB (39.7%) over fixed power relaying and 2.8 dB (47.5%) over fixed gain relaying.

B. Without SD link

Figure 4 plots the SEP of ARGTP relaying as a function of the average relay transmit power for QPSK. The corresponding SEP curves for fixed gain and fixed power relaying are also shown. Now ARGTP relaying outperforms both fixed gain and fixed power AF relaying by an even larger margin. For example, the savings are 4.3 dB (62.8%) over fixed power relaying and 7.7 dB (83.0%) over fixed gain relaying at an SEP of 2×10^{-2}. This is because in the absence of the direct SD link, the relay influences the signal quality at the destination much more.

V. Conclusions

We proposed ARGTP relaying, which generalized the popular fixed power (but variable gain) and fixed gain (but variable power) AF relaying rules. In ARGTP relaying, the relay adapts both its instantaneous transmit power and gain as a function of the channel gains of the links incident on the relay. We derived the SEP-optimal AF relay gain policy and practically amenable closed-form bounds for it. We also analyzed its SEP.

We saw that ARGTP relaying is considerably more energy-efficient than both fixed power and fixed gain relaying. Furthermore, the weaker the direct source-destination link, the more marked is the energy-efficiency. The substantial gains achieved by ARGTP relaying motivate its use in AF cooperative networks. An interesting problem for future work is an analysis of the impact of imperfect channel state information on the performance of ARGTP relaying, and its extension to a system with multiple AF relays.

Appendix

A. Proof of Result 1

The function $\exp\left(-\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\gamma_{rd}+x^{-1}}\right)$ and the power constraint are convex in x. Therefore, using Lagrange multipliers, the problem at hand is equivalent to minimizing $L_\lambda(x)$ for each value of γ_{sr} and γ_{rd}, where

$$L_\lambda(x) \triangleq \exp\left(-\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\gamma_{rd}+x^{-1}}\right) + \lambda\left(x\left(\gamma_{sr}+1\right)-1\right). \quad (20)$$

Note that the expectation operator has been dropped since in the unconstrained formulation x can now be optimized for each γ_{sr} and γ_{rd}. Since $L_\lambda(x)$ is convex in x, the optimal value of x is unique. It is the non-negative solution of

$$\frac{\partial L_\lambda(x)}{\partial x} = -\exp\left(-\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\gamma_{rd}+x^{-1}}\right)\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{(\gamma_{rd}x+1)^2} + \lambda(\gamma_{sr}+1) = 0, \quad (21)$$

if it exists, and is 0, otherwise. Let the non-negative solution be denoted by $x_0(\gamma_{sr}, \gamma_{rd})$. Simplifying (21) results in (12).

The boundary of the region in which $\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd})$ is 0 is obtained by substituting $x = 0$ in (12). Further, it can be verified that $\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd}) = 0$, for all $\gamma_{rd} < \mathcal{B}(\gamma_{sr})$.

B. Proof of Result 2

Using the inequality $\exp(x) \geq 1 + x^2$, for $x \geq 0$, in (12), we get, for $\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})$,

$$\phi(x) \geq (x\gamma_{rd} + 1)^2 \left(1 + \frac{a^2 x^2}{(x\gamma_{rd} + 1)^2}\right) - \frac{a}{\lambda\left(\gamma_{sr} + 1\right)},$$

where $a \triangleq \gamma_{sr}\gamma_{rd} \sin^2\left(\frac{\pi}{M}\right)$. Simplifying further, we get

$$\phi(x) \geq (x\gamma_{rd} + 1)^2 + a^2 x^2 - \frac{a}{\lambda\left(\gamma_{sr} + 1\right)}. \tag{22}$$

Expanding the right side of the above inequality and rearranging terms yields the following quadratic form in x:

$$\phi(x) \geq \left(\gamma_{rd}^2 + a^2\right) x^2 + 2\gamma_{rd} x + 1 - \frac{a}{\lambda\left(\gamma_{sr} + 1\right)} \triangleq \Omega(x). \tag{23}$$

When $\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})$, it can be easily verified that $\Omega(x)$ has exactly one positive root $\beta_u(\gamma_{sr}, \gamma_{rd})$, which is given in (15). Since $\phi(x) \geq \Omega(x)$ and since both are convex for $x \geq 0$, it can be shown that $\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd}) \leq \beta_u(\gamma_{sr}, \gamma_{rd})$.

C. Derivation of Result 3

From (12), $\exp\left(\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\gamma_{rd} + \beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd})^{-1}}\right) = \frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\lambda(\gamma_{sr}+1)}$ $\times \left(\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd})\gamma_{rd} + 1\right)^{-2}$, for $\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})$, and is equal to 1, otherwise. Therefore, the expression for SEP_u above becomes

$$\begin{aligned}\text{SEP}_u = &\frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_0^{\mathcal{B}(\gamma_{sr})} \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr} d\gamma_{rd} \\ &+ \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{\lambda(\gamma_{sr} + 1)}{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)} \left(\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd})\gamma_{rd} + 1\right)^2 \\ &\times \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd}.\end{aligned} \tag{24}$$

Let the first term in the expression for SEP_u above be denoted by I_1. Using [15, (3.324.1)], it can be shown that I_1 simplifies in closed-form to (19).

Since $\beta_{\text{opt}} \leq \beta_u$, it can be shown from (15) that, for $\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})$,

$$\begin{aligned}&\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd})\gamma_{rd} + 1 \leq \beta_u(\gamma_{sr}, \gamma_{rd})\gamma_{rd} + 1 \\ &\leq \frac{\gamma_{sr}^2 \sin^4\left(\frac{\pi}{M}\right) + \sqrt{\frac{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)}{\lambda(\gamma_{sr}+1)} \left(1 + \gamma_{sr}^2 \sin^4\left(\frac{\pi}{M}\right)\right)}}{1 + \gamma_{sr}^2 \sin^4\left(\frac{\pi}{M}\right)}.\end{aligned} \tag{25}$$

Using the above inequality and substituting (19) in (24), we get

$$\begin{aligned}\text{SEP}_u \leq\ &\text{SEP}_{uu} = I_1 \\ &+ \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{\lambda\left(\gamma_{sr} + 1\right)\left(b_1 + \sqrt{b_2\gamma_{rd}}\right)^2}{\gamma_{sr}\gamma_{rd}\sin^2\left(\frac{\pi}{M}\right)} \\ &\times \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd},\end{aligned} \tag{26}$$

where b_1 and b_2 are defined in the result statement. The double integral term in (26), which is denoted by I_2, can be recast as

$$I_2 = \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \mathcal{B}(\gamma_{sr}) \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) I_2^{\text{in}}(\gamma_{sr})\, d\gamma_{sr}, \tag{27}$$

where $I_2^{\text{in}} \triangleq \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{b_1^2 + b_2\gamma_{rd} + 2b_1\sqrt{b_2\gamma_{rd}}}{\gamma_{rd}} \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd}$. Further, I_2^{in} can be split as

$$I_2^{\text{in}} = b_1^2\varphi_1 + b_2\varphi_2 + 2b_1\sqrt{b_2}\varphi_3, \tag{28}$$

where $\varphi_1 \triangleq \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{1}{\gamma_{rd}} \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd} = \text{Ei}\left(\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right)$, $\varphi_2 \triangleq \int_{\mathcal{B}(\gamma_{sr})}^\infty \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd} = \overline{\gamma}_{rd} \exp\left(-\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right)$, and $\varphi_3 \triangleq \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{1}{\sqrt{\gamma_{rd}}} \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd} = \sqrt{\pi\overline{\gamma}_{rd}}\, \text{erfc}\left(\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right)$. Substituting all the above expressions in (26) yields the desired expression in (18).

The derivation of the lower bound expression is skipped due to space constraints, and is available in [17].

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