New Insights into Optimal Discrete Rate Adaptation for Average Power Constrained Single and Multi-Node Systems

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***Abstract*—The throughput-optimal discrete-rate adaptation policy, when nodes are subject to constraints on the average power and bit error rate, is governed by a power control parameter, for which a closed-form characterization has remained an open problem. The parameter is essential in determining the rate adaptation thresholds and the transmit rate and power at any time, and ensuring adherence to the power constraint. We derive novel insightful bounds and approximations that characterize the power control parameter and the throughput in closed-form. The results are comprehensive as they apply to the general class of Nakagami-m ($m \geq 1$) fading channels, which includes Rayleigh fading, uncoded and coded modulation, and single and multi-node systems with selection. The results are appealing as they are provably tight in the asymptotic large average power regime, and are designed and verified to be accurate even for smaller average powers.**

***Index Terms*—Fading channels, power adaptation, rate adaptation, selection, asymptotic analysis, bounds, approximations.**

I. Introduction

RATE and power adaptation is a fundamental technology that underpins several current and next generation wireless systems, and enables spectrally efficient transmission in fading channels. In it, the transmit power, symbol rate, constellation, coding, or any combination thereof are adapted based on channel conditions [1]. Given its importance, considerable attention has been devoted to rate and power adaptation over the past two decades [2]–[9]. The goal has been to determine the rate and transmit power as a function of the channel gain, h, given a constraint on the average transmit power, so as to maximize the throughput.

An information-theoretic analysis of rate adaptation for a single node was developed in [2]. Optimal rate and power adaptation using uncoded M-ary constellations subject to a bit error rate (BER) constraint was studied in [3], [4]. Both continuous rate adaptation and the more practically relevant discrete rate adaptation were studied. The optimal policy when the node is subject to a weaker average BER constraint was characterized in [4]. This was extended to a general class of coded modulation schemes in [5]. The effect of imperfect channel estimates on rate adaptation subject to either instantaneous BER or average BER constraints was analyzed in [7], [8]. Adaptation in multi-node systems in multi-rate code division multiple access (CDMA) systems was considered in [6], with quality of service constraints also being considered in [9].

In both uncoded and coded adaptive modulation schemes, the throughput-optimal discrete rate adaptation policy for a single node, which is subject to an average power constraint and an instantaneous BER constraint, is as follows [3]–[5]. The node transmits with rate r_j and with a power given by $\frac{d_j}{h}$, if $h \in [\eta k_j, \eta k_{j+1})$, where k_j and d_j are constants that depend on the modulation and coding used for rate r_j, for $1 \leq j \leq M$.[1] Here, ηk_j, $1 \leq j \leq M+1$, are the rate adaptation thresholds. The parameter η, which completely determines the optimal adaptation policy, is called the *power control parameter*. It ensures that the average transmit power is constrained to $\bar{P}$ as follows:

$$\sum_{j=1}^{M} d_j \int_{\eta k_j}^{\eta k_{j+1}} \frac{f(h)}{h}\, dh = \bar{P}, \tag{1}$$

where $f(\cdot)$ is the probability density function (PDF) of h.

We thus see that η is a key parameter that drives the optimal adaptation policy. Without it, the transmit rate and power cannot be determined at any time. Its knowledge is also necessary to ensure adherence to the power constraint. Despite its importance, a closed-form expression for η is not available in the literature. Instead, it has to be numerically computed from (1). Similarly, a closed-form expression for the optimal throughput has also remained an open problem.

A similar theoretical framework and a challenging open problem arise when multiple rate-adaptive nodes share a channel to transmit to a common sink, and only a single node with the highest channel gain is selected. Such selection is of practical interest since synchronization among multiple transmit nodes is not required. Even here, the throughput-optimal transmission rule turns out to be similar, with the corresponding power control parameter η being the solution of the following power constraint equation:

$$\sum_{j=1}^{M} d_j \int_{\eta k_j}^{\eta k_{j+1}} \left(F(h)\right)^{K-1} \frac{f(h)}{h}\, dh = \bar{P}, \tag{2}$$

where K is the number of nodes. In this case too, closed-form expressions for η and optimal throughput are not known.

Given the practical importance of discrete rate adaptation in wireless systems today, it is of great interest to analytically

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[1]Note that the optimal transmit power policy is different from channel inversion, which is used in fixed rate systems, because the transmit power depends on the rate chosen through the parameter d_j. In general, d_j increases as r_j increases.

characterize η and optimal throughput, and gain a deeper understanding of their functional dependence on system parameters. The problem is also theoretically relevant because considerable research on link adaptation, including extensions to channel estimation errors and time-varying channels [7], [8], opportunistic scheduling [10], multi-rate CDMA [6], [11], orthogonal frequency division multiplexing (OFDM) [12], and multiple antenna systems [12], [13], has been motivated by the aforementioned model.

A. Contributions

In this paper, we develop novel expressions in the form of bounds and approximations for η and optimal throughput for both single and multi-node rate adaptive systems for the general class of Nakagami-m fading channels, for all $m \geq 1$. Nakagami fading closely approximates line-of-sight (LoS) Ricean fading, includes non-LoS Rayleigh fading as a special case, models L-branch time or spatial diversity, and provides a closer match to some experimental data than several common distributions [14]. It is also extensively used in the literature, e.g., [8], [15], [16]. Thus, our results have wide applicability.

The expressions come with theoretical guarantees, in that they are shown to be tight in the asymptotic regime of large transmit power.[2] They provide new insights into the behavior of η and lead to novel closed-form expressions for both η and throughput. Equally importantly, they are designed and verified to be accurate in a large portion of the non-asymptotic regime, which makes them useful in system design. They are also computationally useful as good initial values for algorithms that compute η numerically. To the best of our knowledge, this is the first time that such closed-form expressions have been derived. An innovative asymptotic approach and several new tricks are used to overcome the challenge of characterizing η – accurately and analytically – even though it appears in the integration limits in the power constraint in (1).

The paper is organized as follows. Sections II and III address the single and multi-node cases, respectively. Simulation results are presented in Sec. IV, and are followed by our conclusions in Sec. V.

II. Single Node Discrete Rate Adaptation: General Framework

Consider a single node that transmits a symbol x over a frequency-flat block-fading channel with a power gain h. The baseband received signal, y, equals

$$y = \sqrt{h}e^{j\phi}x + n,$$

where ϕ is the phase of the channel response and n is additive white Gaussian noise (AWGN) with power spectral density $N_0/2$. Without loss of generality, the average channel power gain is set to unity. The transmitter and receiver are assumed to know the channel gain, as has also been assumed in [3], [4], [17]. This can be achieved in practice through pilots and feedback. The resultant uncoded BER for transmit power P_{tx} and a general M-ary constellation of size μ is given by $c_1 \exp\left(\frac{-c_2 h P_{\text{tx}}}{N_0 B(\mu^{c_3} - c_4)}\right)$, where $c_1, \dots, c_4$ are modulation-specific real constants [4]. The constellation size is chosen from the set $\mathcal{M} = \{\mu_1, \mu_2, \dots, \mu_M\}$, with $\mu_1 = 1$ (no transmission) and $\mu_1 < \mu_2 < \cdots < \mu_M$.

The throughput-optimal policy subject to an instantaneous BER constraint of P_b and an average transmit power constraint of $\bar{P}$ is as follows [4]: The node transmits with a constellation of size μ_j if $h \in [H_j, H_{j+1})$, where $H_1, \dots, H_{M+1}$ are called the *rate adaptation thresholds*. They equal

$$H_j = \eta k_j, \quad \text{for } 1 \leq j \leq M+1. \tag{3}$$

Here, η is a constant, $k_1 = 0$, $k_2 = \frac{1}{c_2}\log_e\left(\frac{c_1}{P_b}\right)\frac{(\mu_2)^{c_3} - c_4}{\log_2(\mu_2)}$, $k_j = \frac{1}{c_2}\log_e\left(\frac{c_1}{P_b}\right)\frac{(\mu_j)^{c_3} - (\mu_{j-1})^{c_3}}{\log_2(\mu_j) - \log_2(\mu_{j-1})}$, for $3 \leq j \leq M$, and $k_{M+1} = \infty$.[3] Further, the transmit power, $P_{\text{tx}}(h, \mu_j) \in \mathbb{R}^+$, of a node that transmits with a constellation of size μ_j when the channel gain is h is

$$P_{\text{tx}}(h, \mu_j) = \frac{d_j}{h}, \quad \text{for } 1 \leq j \leq M+1, \tag{4}$$

where $d_j = \frac{N_0 B}{c_2}\left((\mu_j)^{c_3} - c_4\right)\log_e\left(\frac{c_1}{P_b}\right)$, for $2 \leq j \leq M$, and $d_1 = 0$.

When coded modulation schemes such as coset codes (which includes trellis codes and lattice codes) are used, the optimal policy remains the same. The difference arises in d_j, which depends on the coding gain G of the code [5], [18]. The coding gain model is motivated by Shannon's capacity formula. Thus, the framework applies to both coded and uncoded adaptive modulation.

A. Results about Power Control Parameter and Throughput

We begin with the following simple proposition about η. Its proof follows in a straightforward manner from (1) and is omitted to conserve space.

Proposition 1: Let η_1 and η_2 be the power control parameters corresponding to average transmit powers of $\bar{P}_1$ and $\bar{P}_2$, respectively. Then, $\bar{P}_1 > \bar{P}_2$ if and only if (iff) $\eta_1 < \eta_2$.

Thus, the larger the average transmit power, the smaller is the corresponding value of η. The nature of the relationship between η and $\bar{P}$ is captured below, first for Rayleigh fading and then for Nakagami-m ($m > 1$) fading, by means of novel asymptotically tight bounds.

Proposition 2: For Rayleigh fading and $\eta k_M \leq 4$,

$$\eta \geq \frac{1}{k_M}\exp\left(\alpha_1 - \gamma_0\right)\exp\left(-\frac{\bar{P}}{d_M}\right), \tag{5}$$

where $\alpha_1 = \sum_{j=2}^{M-1}\frac{d_j}{d_M}\log_e\left(\frac{k_{j+1}}{k_j}\right)$ and γ_0 is Euler's constant [19]. Further, the bound is asymptotically tight, *i.e.*, $\lim_{\bar{P}\to\infty}\frac{\eta}{\exp\left(-\frac{\bar{P}}{d_M}\right)} = \frac{1}{k_M}\exp\left(-\gamma_0 + \alpha_1\right)$. The corresponding asymptotically tight bound for the optimal throughput,

[2]For Nakagami-m fading with $m > 1$ or the multi-node case, the maximum average transmit power of the optimal policy turns out to be finite. In this case, the asymptotic regime shall refer to the average transmit power being close to its maximum value.

[3]Thus, $H_{M+1} = \infty$ for any $\eta > 0$. When $\eta = 0$, we define H_{M+1} to be ∞. Similarly, $H_1 = 0$ always.

$\overline{R}^*(\bar{P})$, is given by

$$\overline{R}^*(\bar{P}) \leq \sum_{j=2}^{M} \left[\exp\left(-\frac{k_j}{k_M} \exp\left(-\gamma_0 + \alpha_1 - \frac{\bar{P}}{d_M} \right) \right) \times \log_2 \left(\frac{\mu_j}{\mu_{j-1}} \right) \right]. \quad (6)$$

Proof: The proof is relegated to Appendix A. ■

The above result shows that, for large $\bar{P}$, η decreases exponentially as a function of $\bar{P}$. The dependence of η and the optimal throughput on the system parameters, such as $\{k_j\}$ and $\{d_j\}$, where $\{x_j\}$ denotes $\{x_1, x_2, \ldots\}$, is now very evident. Notice that the bound in (5) is true so long as $\eta k_M \leq 4$; this condition holds even for relatively small $\bar{P}$. For even lower $\bar{P}$, (5) is useful as an approximation.

Proposition 3: For Nakagami-m $(m > 1)$ fading, $\eta = 0$ iff $\bar{P} \geq \bar{P}_{\max}$, where $\bar{P}_{\max} = \frac{m}{m-1} d_M$ is the maximum average power consumed by the optimal policy. For $\bar{P} \leq \bar{P}_{\max}$,

$$\lim_{\bar{P} \to \bar{P}_{\max}} \frac{1}{\eta} \log_e \left[1 - \alpha_2 (\bar{P}_{\max} - \bar{P})^{\frac{1}{m-1}} \right] = -m k_M, \quad (7)$$

where $\alpha_2 = \left(\frac{\Gamma(m)(m-1)}{m \sum_{j=2}^{M} (d_j - d_{j-1}) \left(\frac{k_j}{k_M} \right)} \right)^{\frac{1}{m-1}}$ and $\Gamma(\cdot)$ is the Gamma function [19]. Furthermore, (7), when rearranged, is an upper bound for $m \geq 2$, *i.e.*,

$$\eta \leq -\frac{1}{m k_M} \log_e \left[1 - \alpha_2 (\bar{P}_{\max} - \bar{P})^{\frac{1}{m-1}} \right]. \quad (8)$$

The corresponding result for the throughput, $\overline{R}^*(\bar{P})$, which is an asymptotically tight bound for $m \geq 2$ and is an asymptotically tight approximation for $1 < m < 2$, is as follows:

$$\overline{R}^*(\bar{P}) \geq \log_2(\mu_M) - \frac{1}{\Gamma(m)} \sum_{j=2}^{M} \left[\log_2 \left(\frac{\mu_j}{\mu_{j-1}} \right) \times \gamma \left(m, -\frac{k_j}{k_M} \log_e \left(1 - \alpha_2 (\bar{P}_{\max} - \bar{P})^{\frac{1}{m-1}} \right) \right) \right]. \quad (9)$$

Here, $\gamma(\cdot, \cdot)$ is the lower incomplete gamma function [19].

Proof: The proof is relegated to Appendix B. ■

Note that unlike the case of Rayleigh fading ($m = 1$), the maximum average power consumed for $m > 1$ is finite. The above proposition shows that (8) is an asymptotically tight approximation for η for all $m > 1$. Further, for $m \geq 2$, it refines the result by showing that it is, in fact, an asymptotically tight *upper bound*. Notice how η depends on the difference between $\bar{P}_{\max}$ and $\bar{P}$ and on various system parameters. Note also that Prop. 3 does not apply for $m = 1$ (Rayleigh fading).

III. Multiple Rate Adaptive Nodes With Selection

As shown in Fig. 1, we now consider a system with K (≥ 2) nodes that have data to transmit to a common sink over a flat-fading channel. Let h_i denote the channel power gain between the i^{th} node and the sink; $h_1, \ldots, h_K$ are assumed to be independent and identically distributed. The models for the channel gain, noise, and BER are as stated before for the single node.

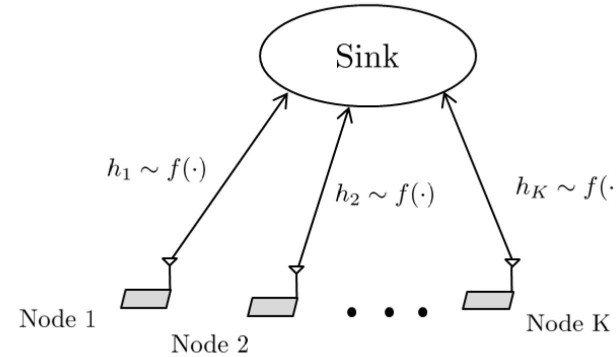

Fig. 1. Illustration of a multi-node system consisting of K rate- and power-adaptive nodes that have data to transmit to a common sink node over a shared channel.

A. Throughput-Optimal Scheme and Results about the Power Control Parameter and Throughput

Maximizing the average throughput requires determining: (i) which node to select, and (ii) the selected node's rate and transmit power. The throughput-optimal adaptation policy is as follows:[4]

- *Selection rule:* The index, s, of the selected node is

$$s = \arg \max_{1 \leq i \leq K} h_i. \quad (10)$$

- *Rate and power adaptation rule:* The selected node, s, transmits with a constellation of size μ_j and transmit power $P_{\text{tx}}(h, \mu_j) = \frac{d_j}{h}$, if $\eta k_j \leq h_s < \eta k_{j+1}$, where $k_1 = 0$, $k_{M+1} = \infty$, and $\{k_j\}$, $2 \leq j \leq M$, are as given in Sec. II.

Here, η is the implicit solution of the following power constraint equation:

$$\bar{P} = \sum_{j=2}^{M-1} d_j \int_{\eta k_j}^{\eta k_{j+1}} \frac{f(h)}{h} (F(h))^{K-1} \, dh + d_M \int_{\eta k_M}^{\infty} \frac{f(h)}{h} (F(h))^{K-1} \, dh. \quad (11)$$

As before, finding closed-form expressions for η has remained an open problem. The following novel results capture the relationship between η and $\bar{P}$, and help solve this problem.

Proposition 4: For Rayleigh fading and $K \geq 2$ nodes, the power control parameter is $\eta = 0$ iff $\bar{P} \geq \bar{P}_{\max}$, where

$$\bar{P}_{\max} = d_M \sum_{i=0}^{K-1} (-1)^{i+1} \binom{K-1}{i} \log_e(i+1), \quad (12)$$

is the maximum average transmit power that is consumed by the optimal policy. Further, the following approximation is asymptotically tight:

$$\eta \approx -\frac{1}{k_M} \log_e \left[1 - x_0 - \frac{\beta x_0^2}{K(1 - \beta x_0) - 1} \right], \quad (13)$$

[4]It can be proved from the results derived in [6]. Given the policy's intuitive form, the proof is skipped to conserve space.

where

$$x_0 \triangleq \left(\frac{(\bar{P}_{\max} - \bar{P})(K-1)}{\sum_{j=2}^{M}(d_j - d_{j-1}) \left(\frac{k_j}{k_M}\right)^{K-1}} \right)^{\frac{1}{K-1}}, \tag{14}$$

$$\beta \triangleq \frac{(K-1)\sum_{j=2}^{M}(d_j - d_{j-1}) \left(\frac{k_j}{k_M}\right)^{K}}{2K\sum_{j=2}^{M}(d_j - d_{j-1}) \left(\frac{k_j}{k_M}\right)^{K-1}}. \tag{15}$$

The corresponding asymptotically tight approximation for the system throughput, $\overline{R}^*(\bar{P})$, is

$$\overline{R}^*(\bar{P}) \approx \log_2 \mu_M - \sum_{j=2}^{M} \log_2 \left(\frac{\mu_j}{\mu_{j-1}}\right) \times \left[1 - \exp\left(\frac{k_j}{k_M} \log_e \left(1 - x_0 - \frac{\beta x_0^2}{K(1-\beta x_0) - 1}\right)\right)\right]^K. \tag{16}$$

Proof: The proof is relegated to Appendix C. ∎

Note that the maximum average power consumed is now finite even for Rayleigh fading, and that η depends on the difference between $\bar{P}_{\max}$ and $\bar{P}$.

Next, we tackle multi-node rate adaptation over Nakagami-m fading with $m > 1$. Again, $\eta = 0$ if $\bar{P} \geq \bar{P}_{\max}$, where $\bar{P}_{\max} = \frac{d_M m^m}{\Gamma^K(m)} \int_0^\infty h^{m-2} \gamma^{K-1}(m, mh) e^{-mh}\, dh$ and $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function [19].

Approximation *1:* For Nakagami-m fading with $m \geq 1$ and $K \geq 2$ nodes, if $\bar{P} < \bar{P}_{\max}$, then

$$\eta \approx -\frac{1}{mk_M} \times \log_e \left[1 - \left(\frac{(\bar{P}_{\max} - \bar{P})(b+1)\Gamma^K(m)}{am^{m-1}\sum_{j=2}^{M}(d_j - d_{j-1}) \left(\frac{k_j}{k_M}\right)^{b+1}}\right)^{\frac{1}{b+1}}\right], \tag{17}$$

where $a = \xi^{m-2}\gamma^{K-1}(m, m\xi)e^{-m\xi}$, $b = e^{m\xi - 1}$, and ξ is a constant that satisfies:

$$\frac{m-2}{\xi} + \frac{(K-1)m^m \xi^{m-1} e^{-m\xi}}{\gamma(m, m\xi)} = m. \tag{18}$$

The corresponding expression for the throughput is

$$\overline{R}^*(\bar{P}) \approx \log_2(\mu_M) - \frac{1}{(\Gamma(m))^K} \sum_{j=2}^{M} \log_2 \left(\frac{\mu_j}{\mu_{j-1}}\right) \times \left[\gamma\left(m, -\frac{k_j}{k_M} \log_e \left(1 - \omega^{\frac{1}{b+1}}\right)\right)\right]^K, \tag{19}$$

where $\omega = \frac{(\bar{P}_{\max} - \bar{P})(b+1)(\Gamma(m))^K}{am^{m-1}\sum_{i=2}^{M}(d_i - d_{i-1})\left(\frac{k_i}{k_M}\right)^{b+1}}$.

Derivation: The derivation is relegated to Appendix D.

The main advantage of the approach above is that ξ and, thus, a and b do not depend on $\bar{P}$. Values of ξ for different values of m and K are tabulated in Table I. The values of a and b and, hence, η are then readily computed. Note that this is computationally less expensive than solving (11) numerically for η for different $\bar{P}$. As we shall see in Sec. IV, the error in the approximation in (17) is negligible.

TABLE I

Multiple nodes, Nakagami-m ($m > 1$) fading: Table of values of ξ for different values of Nakagami-m fading parameter (m) and number of nodes (K).

Nakagami fading parameter (m)	*Number of nodes (K)*	ξ
2	5	1.33
	10	1.81
	15	2.07
3	5	1.34
	10	1.7
	15	1.9
4	5	1.33
	10	1.63
	15	1.79
5	5	1.32
	10	1.57
	15	1.71

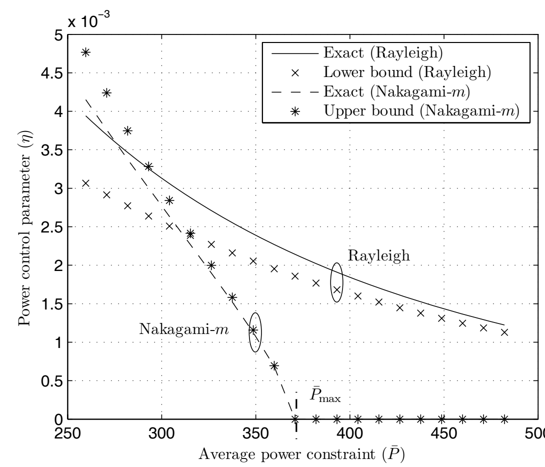

Fig. 2. Single node: power control parameter, η, as a function of average transmit power, $\bar{P}$.

Note that in the multi-node case, both Prop. 4 and Approx. 1 apply for $m = 1$ (Rayleigh fading). However, Prop. 4 is simpler to use for Rayleigh fading.

IV. Simulation Results on Accuracy in Asymptotic/Non-Asymptotic Regimes

We illustrate our results for M-QAM with the set of constellation sizes given by $\mathcal{M} = \{1, 4, 16, 64\}$. The average transmit power is normalized with respect to $N_0 B$, where B is the bandwidth. We first consider the single node case. Figure 2 plots the exact value of η and its bounds as a function of $\bar{P}$, for both Rayleigh and Nakagami-m ($m = 2.5$) fading.

Figure 3 plots the exact value and the bounds for the corresponding optimal throughput. We observe that the bounds on η and throughput are asymptotically tight. Further, they are quite accurate even for relatively small values of $\bar{P}$. For example, the error between the bound and the exact value of η is 11% at $\bar{P} = 400$ for $m = 1$ and is 7% at $\bar{P} = 0.8\bar{P}_{\max}$ for $m = 2.5$. The corresponding errors for the throughput are just 1.9% and 0.6%.

We now consider the multi-node case with $K = 10$ nodes. Figure 4 plots the exact and the approximate values of the power control parameter, η, as a function of the average

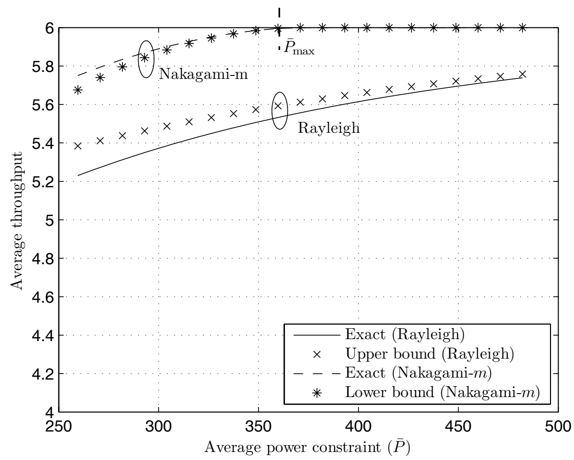

Fig. 3. Single node: Zoomed-in view of throughput as a function of average transmit power.

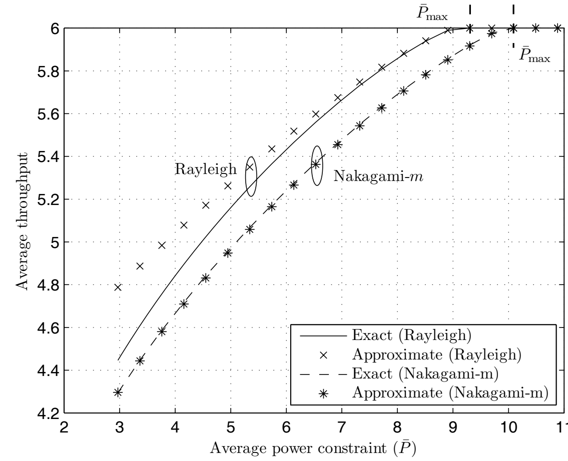

Fig. 5. Multiple nodes: Zoomed-in view of throughput as a function of average transmit power per node (K = 10).

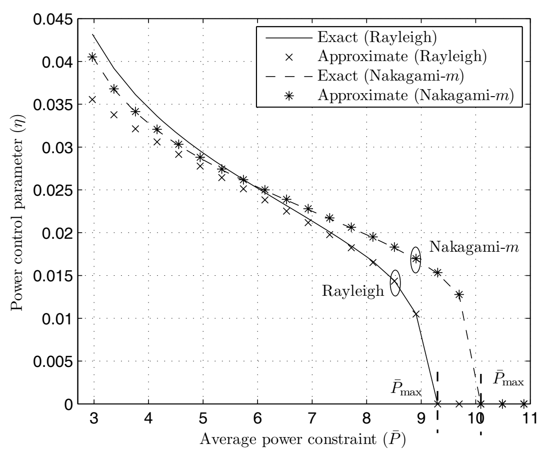

Fig. 4. Multiple nodes: Power control parameter (η) as a function of average transmit power per node (K = 10).

power for both Rayleigh and Nakagami-m ($m = 1.5$) fading. The values of $\bar{P}_{\max}$ are lower compared to the single node case. We observe that the approximations are tight and are quite accurate even for relatively small values of $\bar{P}$. For example, the error between the approximate and exact values is just 3.4% and 0.4% for Rayleigh and Nakagami-m fading, respectively, when $\bar{P}$ is 66% of the corresponding $\bar{P}_{\max}$. Figure 5 plots the corresponding curves for throughput. As before, the error between the approximate and exact values is very small. At $\bar{P} = 0.66\bar{P}_{\max}$, the error is just 0.9% and 0.1% for $m = 1$ and $m = 1.5$, respectively.

V. Conclusions

The optimal power and discrete rate adaptation policy for a wireless system, which consists of one or more nodes that are subject to constraints on the average power and the instantaneous BER, is governed by the power control parameter, η. The power control parameter plays a central role in the rate adaptation policy as it directly determines the rate adaptation thresholds. Without knowing it, the transmitter cannot determine its transmit rate and power at any time instant, nor can it ensure adherence to the power constraint. However, this important parameter has thus far been computed only numerically.

We showed, for the first time, that closed-form asymptotically tight bounds or approximations for η as well as optimal throughput are indeed possible. Furthermore, the novel expressions were verified to be accurate even in a large portion of the non-asymptotic average power regime. For example, the error in throughput when $\bar{P}$ is 66% of $\bar{P}_{\max}$ was less than 1% for a system with 10 nodes. New tricks were developed to overcome the challenge of analytically characterizing η, despite it occurring in the limits of integration of several integrals in the average power constraint equation. Altogether, our comprehensive analysis covers the general and widely applicable class of Nakagami-m ($m > 1$) and Rayleigh fading channels, and handles both uncoded and coded modulation, and single and multi-node systems with selection. The expressions brought out the functional dependence of η and the optimal throughput on system parameters such as the set of discrete rates, modulation schemes, and average power.

Given the fundamental importance of link adaptation in current and next generation wireless systems, our results motivate the investigation of similar results in several other rate-adaptive systems that use multiple antenna technology, multi-rate CDMA, or OFDM, or use average BER constraints, discrete power adaptation, and in systems with imperfect channel estimates and time-varying channels. Further, it would be interesting to attempt to derive universal bounds or approximations that are tight for all possible values of the average power and for fading models other than Nakagami-m fading.

Appendix

A. *Proof of Prop. 2*

Substituting $f(h) = e^{-h}$, for $h > 0$, in (1), we get

$$\bar{P} = \sum_{j=2}^{M} d_j \int_{\eta k_j}^{\eta k_{j+1}} \frac{e^{-h}}{h}\, dh,$$

$$= d_M E_1(\eta k_M) + \sum_{j=2}^{M-1} d_j \left(E_1(\eta k_j) - E_1(\eta k_{j+1})\right), \quad (20)$$

where $E_1(\cdot)$ is the Euler exponential integral [19]. Substituting $E_1(x) = -\gamma_0 - \log_e(x) + \sum_{l=1}^{\infty} \frac{(-1)^{l+1}x^l}{l \cdot l!}$, where γ_0 is Euler's constant [19], in (20) gives

$$\frac{\bar{P}}{d_M} = -\gamma_0 - \log_e(k_M \eta) + \sum_{j=2}^{M-1} \frac{d_j}{d_M} \log_e\left(\frac{k_{j+1}}{k_j}\right) + \sum_{j=2}^{M} \frac{(d_j - d_{j-1})}{d_M} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l \cdot l!} (\eta k_j)^l. \quad (21)$$

For $\eta k_M \leq 4$, we can show that $\frac{(\eta k_j)^{l+1}}{(l+1)\cdot(l+1)!} \leq \frac{(\eta k_j)^l}{l \cdot l!}$, $\forall\, l \geq 1$ and $2 \leq j \leq M$. Thus, the above alternating series in l is bounded below by 0. Since $d_j > d_{j-1}$, it follows that $\frac{\bar{P}}{d_M} \geq -\gamma_0 - \log_e(\eta k_M) + \sum_{j=2}^{M-1} \frac{d_j}{d_M} \log_e\left(\frac{k_{j+1}}{k_j}\right)$. Taking exponentials on both sides and simplifying gives (5). Taking the limit as $\bar{P} \to \infty$ (which is equivalent to $\eta \to 0$) in (21) yields the desired limit result.

The optimum throughput simplifies as follows:

$$\overline{R}^*(\bar{P}) = \sum_{j=2}^{M} \log_2(\mu_j) \int_{\eta k_j}^{\eta k_{j+1}} f(h)\, dh,$$
$$= \log_2(\mu_M) - \sum_{j=2}^{M} F(\eta k_j) \log_2\left(\frac{\mu_j}{\mu_{j-1}}\right). \quad (22)$$

Substituting $F(x) = 1 - e^{-x}$, for $x \geq 0$, and the expression for η from (5) into (22) and simplifying yields (6).

B. Proof of Prop. 3

From Prop. 1, maximum average power is consumed iff $\eta = 0$. Substituting $\eta = 0$ and $f(h) = \frac{m^m}{\Gamma(m)} h^{m-1} e^{-mh}$, for $h \geq 0$, in (1), we get

$$\bar{P}_{\max} = \frac{m^m d_M}{\Gamma(m)} \int_0^{\infty} h^{m-2} e^{-mh}\, dh = \frac{m}{m-1} d_M.$$

Now, consider the case where $\bar{P} \leq \bar{P}_{\max}$ and $m \geq 2$. From (1), we have

$$\bar{P} = \frac{m^m}{\Gamma(m)} \sum_{j=2}^{M} d_j \int_{\eta k_j}^{\eta k_{j+1}} h^{m-2} e^{-mh}\, dh,$$
$$= \bar{P}_{\max} - \frac{m^m}{\Gamma(m)} \sum_{j=2}^{M} u_j (d_j - d_{j-1}), \quad (23)$$

where $u_j \triangleq \int_0^{\eta k_j} h^{m-2} e^{-mh}\, dh$. The following two asymptotically tight bounds are the key steps that address the challenge of ferreting η out of the limits of integration: $h \geq \frac{1-e^{-mh}}{m}$ and $\frac{1-e^{-m\eta k_j}}{1-e^{-m\eta k_M}} \geq \frac{k_j}{k_M}$, for $h \geq 0$. Using these, we get

$$u_j \geq \frac{\left(1 - e^{-m\eta k_j}\right)^{m-1}}{(m-1)m^{m-1}} \geq \frac{(1 - e^{-m\eta k_M})^{m-1}}{(m-1)m^{m-1}} \left(\frac{k_j}{k_M}\right)^{m-1}. \quad (24)$$

Substituting (24) in (23) gives

$$\bar{P}_{\max} - \bar{P} \geq \frac{m(1 - e^{-m\eta k_M})^{m-1}}{(m-1)\Gamma(m)} \sum_{j=2}^{M} (d_j - d_{j-1}) \left(\frac{k_j}{k_M}\right)^{m-1},$$

which leads to (8).

Also note that the first inequality in (24) gets reversed for $m < 2$, as a result of which (8) is not a bound on η for $1 < m < 2$. However, from (23), $\frac{\bar{P}_{\max} - \bar{P}}{\sum_{j=2}^{M} u_j (d_j - d_{j-1})} = \frac{m^m}{\Gamma(m)}$, for any $m > 1$. Taking limits as $\bar{P} \to \bar{P}_{\max}$, *i.e.*, $\eta \to 0$, and using $\lim_{h \to 0} \frac{h}{\left(\frac{1-e^{-mh}}{m}\right)} = 1$ and $\lim_{\eta \to 0} \left(\frac{1-e^{-m\eta k_j}}{1-e^{-m\eta k_M}}\right) = \frac{k_j}{k_M}$ results in (7).

The corresponding throughput expression in (9) is obtained by substituting $F(x) = \frac{\gamma(m,x)}{\Gamma(m)}$, $x \geq 0$, and η from (8) in (22).

C. Proof of Prop. 4

We shall use the following notation: *For any two functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ defined on $\mathbb{R}$, we say that $\psi_1(x) \sim \psi_2(x)$ iff* $\lim_{x \to 0} \frac{\psi_1(x)}{\psi_2(x)} = 1$. As in Prop. 1, it can be shown that maximum average power is consumed iff $\eta = 0$. Substituting $f(h) = e^{-h}$ and $\eta = 0$ in (11), we get

$$\bar{P}_{\max} = d_M \int_0^{\infty} \frac{e^{-h}}{h} (1 - e^{-h})^{K-1}\, dh,$$
$$= d_M \lim_{x \to 0} \sum_{i=0}^{K-1} (-1)^i \binom{K-1}{i} E_1((i+1)x). \quad (25)$$

Substituting $E_1(x) = -\gamma_0 - \log_e(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n \cdot n!}$ in the above equation gives the desired expression for $\bar{P}_{\max}$.

Substituting $f(h) = e^{-h}$ in (11) and using (25) yields

$$\bar{P}_{\max} - \bar{P} = \sum_{j=2}^{M} (d_j - d_{j-1}) \int_0^{\eta k_j} \frac{e^{-h}}{h} \left(1 - e^{-h}\right)^{K-1} dh,$$
$$= \sum_{j=2}^{M} (d_j - d_{j-1}) v_j, \quad (26)$$

where $v_j = \int_0^{\eta k_j} \frac{e^{-h}}{h} \left(1 - e^{-h}\right)^{K-1} dh$. The following simple observation is key to simplifying v_j in a manner that, as we shall see, enables η to be approximated in closed-form:

$$\frac{1 - e^{-h}}{h} \sim \frac{1 + e^{-h}}{2}. \quad (27)$$

Thus,

$$v_j \sim \int_0^{\eta k_j} e^{-h} \left(\frac{1 + e^{-h}}{2}\right) \left(1 - e^{-h}\right)^{K-2} dh,$$
$$= \frac{t_j^{K-1}}{K-1} - \frac{t_j^K}{2K}, \quad (28)$$

where $t_j \triangleq 1 - e^{-\eta k_j}$. Substituting this in (26), we get,

$$\bar{P}_{\max} - \bar{P} \sim \sum_{j=2}^{M} (d_j - d_{j-1}) \left(\frac{t_j^{K-1}}{K-1} - \frac{t_j^K}{2K}\right), \quad (29)$$
$$\sim t_M^{K-1} \sum_{j=2}^{M} \frac{d_j - d_{j-1}}{K-1} \left(\frac{t_j}{t_M}\right)^{K-1}. \quad (30)$$

Another key observation is that $\frac{t_j}{t_M} \sim \frac{k_j}{k_M}$. This results in

$$\bar{P}_{\max} - \bar{P} \sim \frac{t_M^{K-1}}{K-1} \sum_{j=2}^{M} (d_j - d_{j-1}) \left(\frac{k_j}{k_M}\right)^{K-1}.$$

Rearranging terms, we get $t_M \sim x_0$ (cf. (14)). Therefore,

$$\eta = -\frac{1}{k_M}\log_e(1-t_M) \sim -\frac{1}{k_M}\log_e(1-x_0). \quad (31)$$

This proves the asymptotic tightness of (13) since x_0 dominates the term containing x_0^2 as $x_0 \to 0$. A perturbation analysis on the roots of the polynomial equation in (29) results in the inclusion of the term containing x_0^2, which refines the result in (31) and makes it more accurate for smaller $\bar{P}$. The intermediate steps are omitted due to space constraints.

The optimal throughput simplifies as follows:

$$\begin{aligned}\overline{R}^*(\bar{P}) &= \sum_{j=2}^{M}\log_2(\mu_j)\int_{\eta k_j}^{\eta k_{j+1}} f(h)\,(F(h))^{K-1}\,dh,\\ &= \log_2(\mu_M) - \sum_{j=2}^{M}(F(\eta k_j))^K \log_2\left(\frac{\mu_j}{\mu_{j-1}}\right). \quad (32)\end{aligned}$$

Substituting $F(x) = 1 - e^{-x}$, $x \geq 0$, and the expression for η from (13) into (32) yields (16).

D. Derivation of Approx. 1

Substituting $f(h) = \frac{m^m}{\Gamma(m)}h^{m-1}e^{-mh}$ and $F(h) = \frac{\gamma(m,mh)}{\Gamma(m)}$, for $h \geq 0$, in (11), we get

$$\begin{aligned}\bar{P} &= \frac{m^m}{\Gamma^K(m)}\sum_{j=2}^{M} d_j \int_{\eta k_j}^{\eta k_{j+1}} h^{m-2}\gamma^{K-1}(m,mh)e^{-mh}\,dh,\\ &= \bar{P}_{\max} - \frac{m^m}{\Gamma^K(m)}\sum_{j=2}^{M} w_j(d_j - d_{j-1}), \quad (33)\end{aligned}$$

where $w_j \triangleq \int_0^{\eta k_j} h^{m-2}\gamma^{K-1}(m,mh)e^{-mh}\,dh$. Let $g_1(h) = h^{m-2}\gamma^{K-1}(m,mh)e^{-mh}$ and $g_2(h) = a(1-e^{-mh})^b e^{-mh}$, where $a, b \in \mathbb{R}^+$. Notice that $g_1(0) = g_2(0) = 0$ and $\lim_{h\to\infty} g_1(h) = \lim_{h\to\infty} g_2(h) = 0$. Further, both $g_1(\cdot)$ and $g_2(\cdot)$ have unique maxima. Let the maximum of $g_1(h)$ occur at $h = \xi$. A key step is to approximate $g_1(h)$ with $g_2(h)$, both of which tend to the same value of 0 for $h \to 0$ as well as $h \to \infty$. For this, a and b are chosen to make $g_2(h)$ have the same maximum as $g_1(h)$ and at $h = \xi$. Thus, ξ is a solution of $\frac{dg_1(h)}{dh}\big|_{h=\xi} = 0$, which gives (18). Using $\frac{dg_2(h)}{dh}\big|_{h=\xi} = 0$ and $g_2(\xi) = g_1(\xi)$ gives $a = \xi^{m-2}\gamma^{K-1}(m,m\xi)e^{-m\xi}$ and $b = e^{m\xi - 1}$.

Replacing $g_1(h)$ with $g_2(h)$ gives

$$w_j \approx \int_0^{\eta k_j} a(1-e^{-mh})^b e^{-mh}\,dh = \frac{a(1-e^{-m\eta k_j})^{b+1}}{m(b+1)}.$$

Substituting this in (33), we get

$$\begin{aligned}\bar{P}_{\max} - \bar{P} &\approx \frac{m^m}{\Gamma^K(m)}\sum_{j=2}^{M}(d_j - d_{j-1})\frac{a(1-e^{-m\eta k_j})^{b+1}}{m(b+1)},\\ &\approx \frac{am^{m-1}}{\Gamma^K(m)(b+1)}(1-e^{-m\eta k_M})^{b+1}\\ &\quad\times \sum_{j=2}^{M}(d_j - d_{j-1})\left(\frac{k_j}{k_M}\right)^{b+1}. \quad (34)\end{aligned}$$

Rearranging terms in the above equation yields (17).

The throughput expression in (19) is obtained by substituting (17) and $F(x) = \frac{\gamma(m,x)}{\Gamma(m)}$, $x \geq 0$, into (32).

References

[1] A. J. Goldsmith, *Wireless Communications* Cambridge University Press, 2005.

[2] A. J. Goldsmith and P. P. Varaiya, "Capacity of fading channels with channel side information," *IEEE Trans. Inf. Theory*, vol. 43, pp. 1986–1992, Nov. 1997.

[3] A. J. Goldsmith and S.-G. Chua, "Variable-rate variable-power M-QAM for fading channels," *IEEE Trans. Commun.*, vol. 45, pp. 1218–1230, Oct. 1997.

[4] S. T. Chung and A. J. Goldsmith, "Degrees of freedom in adaptive modulation: a unified view," *IEEE Trans. Commun.*, vol. 49, pp. 1561–1571, Sep. 2001.

[5] A. J. Goldsmith and S.-G. Chua, "Adaptive coded modulation for fading channels," *IEEE Trans. Commun.*, vol. 46, pp. 595–602, May 1998.

[6] S. A. Jafar and A. Goldsmith, "Optimal rate and power adaptation for multirate CDMA," in *Proc. 2000 VTC Fall*, pp. 994–1000.

[7] S. Falahati, A. Svensson, T. Ekman, and M. Sternad, "Adaptive modulation systems for predicted wireless channels," *IEEE Trans. Commun.*, vol. 52, pp. 307–316, Feb. 2004.

[8] A. Olfat and M. Shikh-Bahaei, "Optimum power and rate adaptation for MQAM in Rayleigh flat fading with imperfect channel estimation," *IEEE Trans. Veh. Technol.*, vol. 57, pp. 2622–2627, July 2008.

[9] D. I. Kim, E. Hossain, and V. K. Bhargava, "Dynamic rate and power adaptation for provisioning class-based QoS in cellular multirate WCDMA systems," *IEEE Trans. Wireless Commun.*, vol. 3, pp. 1590–1601, Sep. 2004.

[10] M. J. Hossain, M.-S. Alouini, and V. K. Bhargava, "Rate adaptive hierarchical modulation-assisted two-user opportunistic scheduling," *IEEE Trans. Wireless Commun.*, vol. 6, pp. 2076–2085, June 2007.

[11] D. I. Kim, E. Hossain, and V. K. Bhargava, "Downlink joint rate and power allocation in cellular multirate WCDMA systems," *IEEE Trans. Wireless Commun.*, vol. 2, pp. 69–80, Feb. 2003.

[12] S. Catreux, V. Erceg, D. Gesbert, and R. W. Heath, "Adaptive modulation and MIMO coding for broadband wireless data networks," *IEEE Commun. Mag.*, pp. 108–115, June 2002.

[13] U. Fernandez-Plazaola, E. Martos-Naya, J. F. Paris, and A. J. Goldsmith, "Adaptive modulation for MIMO systems with channel prediction errors," *IEEE Trans. Wireless Commun.*, vol. 9, pp. 2516–2527, Aug. 2010.

[14] M. Simon and M.-S. Alouini, *Digital Communication over Fading Channels*, 2nd edition. Wiley-Interscience, 2005.

[15] J. E. Garzás, C. B. n. Calzón, and A. G. Armada, "An energy-efficient adaptive modulation suitable for wireless sensor networks with SER and throughput constraints," *EURASIP J. Wireless Commun. Netw.*, vol. 2007.

[16] W. L. Huang and K. B. Letaief, "Cross-layer scheduling and power control combined with adaptive modulation for wireless ad hoc networks," *IEEE Trans. Commun.*, vol. 55, pp. 728–739, Apr. 2007.

[17] A. Gjendemsjø, G. E. Oien, H. Holm, M.-S. Alouini, D. Gesbert, K. J. Hole, and P. Orten, "Rate and power allocation for discrete-rate link adaptation," *EURASIP J. Wireless Commun. Netw.*, vol. 2008, pp. 16:1–16:11.

[18] T. Starr, J. M. Cioffi, and P. J. Silverman, *Understanding Digital Subscriber Line Technology*, 1st edition. Prentice Hall PTR, 1999.

[19] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th edition. Dover, 1972.