# Feedback Overhead-Aware, Distributed, Fast, and Reliable Selection

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***Abstract*—In a communication system in which $K$ nodes communicate with a central sink node, the following problem of selection often occurs. Each node maintains a preference number called a metric, which is not known to other nodes. The sink node must find the 'best' node with the largest metric. The local nature of the metrics requires the selection process to be distributed. Further, the selection needs to be fast in order to increase the fraction of time available for data transmission using the selected node and to handle time-varying environments. While several selection schemes have been proposed in the literature, each has its own shortcomings. We propose a novel, distributed selection scheme that generalizes the best features of the timer scheme, which requires minimal feedback but does not guarantee successful selection, and the splitting scheme, which requires more feedback but guarantees successful selection. The proposed scheme introduces several new ideas into the design of the timer and splitting schemes. It explicitly accounts for feedback overheads and guarantees selection of the best node. We analyze and optimize the performance of the scheme and show that it is scalable, reliable, and fast. We also present new insights about the optimal timer scheme.**

***Index Terms*—Multiple access, multiuser diversity, selection, splitting scheme, timer backoff, throughput, feedback overhead.**

## I. Introduction

IN a communication systems in which $K$ nodes communicate with a central sink node, the following problem often occurs. A random variable (RV) is associated with each node. This RV is called *a metric*. The metrics are local in the sense that they are evaluated based on information local to the node, and their realizations are not known to other nodes. The central node needs to select the 'best' node with the largest metric.

The above metric-based selection problem arises in several wireless systems, some of which are discussed below. In the downlink of a cellular system, the base station needs to transmit to the mobile user with the largest signal-to-noise ratio (SNR) in order to exploit multi-user diversity [2, Chap. 14]. In this case, the metric of a node is its downlink SNR. On the other hand, in case a proportional fair scheduler is employed by the base station (BS), the metric is the ratio $\mathrm{SNR}/\mathbf{E}[\mathrm{SNR}]$, where $\mathbf{E}[\cdot]$ denotes expectation [3]. Other notions of fairness such as max-min fairness [4] and CDF-based fairness [5] can also be formulated as metric-based selection problems by suitably defining the metric. In an amplify-and-forward cooperative relaying system with a source, $K$ relays, and a destination, the source needs to select the relay with the largest value of the harmonic mean of the source-to-relay and relay-to-destination channel gains in order to maximize the SNR at the destination [6], [7].

Selection is practically appealing because it eliminates the problem of ensuring tight symbol-level synchronization among simultaneously transmitting nodes. Relay selection has also been shown to reduce energy consumption [8] and has been also considered for decode-and-forward [7] and compress-and-forward [9] protocols. In a sensor network, selecting the best node for transmission improves network lifetime [10], [11].

### A. Selection Schemes

Given its relevance, the selection problem has been well studied in the literature, and several schemes have been proposed.

A centralized scheme for selection is *polling*. In it, each node sequentially reveals its metric to the sink, which then selects the best node. However, polling is not scalable since the time taken by it to select grows linearly with the number of nodes. This limits the Doppler spread that the system can handle and reduces the fraction of time available for data transmission.

The following two different approaches successfully address the problem of distributed selection with the help of a sink node that coordinates the selection process.

*1) Splitting-based selection [12], [13]:* In it, all nodes whose metrics lie in between two thresholds transmit in a time slot. In each slot, the sink feeds back an idle outcome, if no node transmitted (no signal), success outcome if one node transmitted (decodable signal), and collision outcome if multiple nodes transmitted and interfered with each other (undecodable signal). The thresholds are updated based on this feedback. The splitting scheme is guaranteed to select the best node. For this, it requires less than 2.52 slots, on average, even for the worst case of an asymptotically large number of nodes [12]. Its speed of selection and scalability make it quite appealing.

*2) Single-stage Timer Backoff-based Selection [7], [11], [14], [15]:* In it, every node sets a timer as a function of its metric. When its timer expires, the node transmits a contention packet to the sink. The metric-to-timer mapping is such that the timer of the best node always expires first. However, the scheme will fail to select the best node if, for example, the timer of another node expires within a duration $\Delta$ of the expiry of the timer of the best node [16].

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Different metric-to-timer mappings have been proposed in the literature [7], [11], [14], [15]. In [14], an optimal metric-to-timer mapping that maximizes the probability of success given a maximum selection duration $T_{\max}$ is derived. The optimal mapping turns out to have a special structure – in it, the timers can either expire only at $0, \Delta, \ldots, N\Delta$, where $N = \left\lfloor \frac{T_{\max}}{\Delta} \right\rfloor$ and $\lfloor \cdot \rfloor$ denotes the floor function, or not expire at all. We shall refer to $N$ as the *number of timer levels* henceforth.

*3) Comparisons and Open Issues:* The splitting and timer schemes have their advantages and drawbacks. The splitting scheme is fast when measured in terms of the number of slots required on average to select the best node. However, it requires feedback in every slot, which increases the duration of a slot and, thus, the total time required to select the best node. Therefore, modifying the splitting scheme to reduce its average selection time after accounting for its feedback overheads is a relevant problem.

Unlike the splitting scheme, the timer scheme does not require any feedback, except for a broadcast at the end by the sink to inform which node has been selected. However, the timer scheme cannot guarantee selection of the best node. It is, therefore, of interest to integrate the timer scheme with an appropriate collision resolution mechanism and, thereby, guarantee selection of the best node. In [17], $T_{\max}$ is doubled every time a selection failure occurs. Instead, in [18], the metrics are randomized in case of a failure. However, such randomization degrades overall system performance since the best node may not get selected. Also, the selection may not even be fast.

### B. Contributions

In this paper, we propose a novel selection scheme that is different from the approaches pursued in the literature. It is scalable and is faster than the splitting scheme when the feedback overhead is accounted for, and, unlike the timer scheme, guarantees selection of the best node. It combines the best features of the splitting scheme, namely, fast contention resolution and scalability, and the timer scheme, namely, low feedback.

The scheme runs as follows. First, the single-stage optimal timer scheme is run. If a collision occurs, then the timer scheme is rerun, but with its parameters suitably adjusted. Only the best node and the other nodes that collided with it rescale their metrics and participate in the next stage. Similarly, if no timer expires, then all the nodes rescale their metrics and the timer scheme is rerun. This process is repeated until a success occurs. Each timer stage is followed by the feedback of an idle, success, or collision outcome by the sink to enable the nodes to rescale their metrics.

Several new ideas are introduced in the proposed scheme compared to the splitting and timer schemes. First, the number of timer levels in each timer stage depends on the outcomes. Second, the discrete structure of the single-stage optimal timer scheme, which was proved in [14], is exploited by the proposed scheme to obtain finer information about the time at which the contention packet of the best node is involved in a collision. Third, our approach effectively introduces contention resolution in the timer scheme.

We develop analytical expressions for the average time required by the scheme to select the best node and minimize it. We also derive several new properties about the single-stage optimal timer scheme [14], which are useful in the design and analysis of the proposed scheme. The optimal parameter settings reveal that the number of timer levels should be decreased once a collision occurs. This is contrary to the timer window doubling approach of [17]. The optimal number of timer levels in each stage is shown to be proportional to the square root of the ratio of feedback duration $F$ and $\Delta$.

We also present an extensive set of results that benchmark the net throughputs achieved by the proposed scheme and several other schemes considered in the literature such as splitting, single-stage timer, polling, threshold-based random access [18], and O-CSMA/CA [17]. The net throughgput comparison correctly accounts for all the time overheads of the selection process. It is pertinent because different schemes adopt different compromises between spending time on selecting the best node, settling for a node that is not the best node, and spending the available time for transmitting data using the selected node. The results show that the proposed scheme achieves the highest net throughput among all the benchmark schemes and requires markedly fewer feedback messages than splitting and O-CSMA/CA.

The paper is organized as follows. Section II describes the system model and the single-stage optimal timer scheme. We develop the proposed selection scheme in Section III. Numerical results and performance comparisons are presented in Section IV. Our conclusions follow in Section V. Several mathematical derivations are relegated to the Appendix.

## II. System Model and Timer-Based Selection

We shall use the following notation henceforth. $\mathbb{R}$ denotes the set of all real numbers. The sum $\sum_{i=n}^{m}$ is defined to be zero for $m < n$. The ceil function is denoted by $\lceil \cdot \rceil$. The probability of an event $A$ is denoted by $\mathbf{Pr}[A]$. The expectation of an RV $X$ is denoted by $\mathbf{E}[X]$. The conditional expectation of an RV $X$, conditioned over an event $\mathcal{A}$, is denoted by $\mathbf{E}[X|\mathcal{A}]$. The notation $X \sim U[a, b)$ shall indicate that the RV $X$ is uniformly distributed over the interval $[a, b)$.

As mentioned, we consider a system that consists of $K$ nodes and a sink. Each node $i$ knows its metric $\mu_i$, which is not known to any other node. The goal is to find the best node $i^*$, where $i^* = \mathrm{argmax}_{i \in \{1,2,\ldots,K\}} \mu_i$. The metrics are assumed to be independent and identically distributed (i.i.d.) across nodes. The independence of $\mu_i$s is motivated by the fact that the metrics are typically a function of local channel gains, which decorrelate across a distance of a wavelength [2, Chap. 3]. Assuming that the metrics are statistically identical ensures analytical tractability and is commonly assumed in the selection literature, e.g., [7], [11], [12], [15], [17], [18].

The nodes are assumed to know $K$ and the continuous cumulative distribution function (CDF) of $\mu_i$, $C(\cdot)$, as has also been assumed in [12], [17]–[19]. Physically, this is possible because the number of nodes and the CDF of the metric vary at a time scale that is several orders of magnitude slower than channel fading [2]. For example, the BS in a cellular system or the access point in a wireless local area network (WLAN) can keep track of the number of nodes it is serving and can broadcast it periodically to the nodes.[1] Therefore, without loss of generality, $\mu_i$ is assumed to be uniformly distributed over $[0,1)$, i.e., $\mu_i \sim U[0,1)$. This is because the transformation $\widetilde{\mu}_i = C(\mu_i)$ ensures that $\widetilde{\mu}_i \sim U[0,1)$ [21].

We also assume that a transmission by a single node always succeeds, whereas, when two or more nodes transmit simultaneously, a collision occurs and the receiver cannot decode any of the transmissions, as has also been assumed in [12], [13], [22], [23]. Further, the feedback packet transmitted by the sink is assumed to be received reliably by the nodes. In practice, this is ensured by allowing for a sufficient fading margin in the link budget of the system, which is feasible given the small payload of the contention and feedback packets.

### A. Single-Stage Timer Scheme Basics

In the single-stage timer scheme, each node $i$ sets a timer $T_i = f(\mu_i)$, where $f : [0,1) \to [0, T_{\max}]$ is the metric-to-timer mapping. When the timer expires at time $T_i$, the node transmits a contention packet to the sink communicating its identity. *Using a monotonically non-increasing deterministic mapping ensures that the timer of the best node always expires first*.

The sink, thus, just waits for the first packet to reach it, which by design is from the best node. Therefore, at the end of the selection duration $T_{\max}$, it broadcasts a *feedback message* to all the nodes informing them which node has been selected. We denote $F$ to be the duration from the end of the timer stage, $T_{\max}$, to the instant at which each node correctly receives the feedback message. If the timers of the best node and the second best node expire within a duration $\Delta$ of each other, then the sink fails to decode the packet from the best node, and the single-stage timer scheme fails. Here, $\Delta$ is called the *vulnerability window* [7], [16].

Both $\Delta$ and $F$ are system-dependent parameters. Broadly speaking, their values depend on which of the following two categories the system belongs to; both are of interest to us [24], [25].

- *No hidden nodes and each node can do carrier-sensing:* In such a system, $\Delta$ is a sum of the maximum propagation delay between the nodes, the switching time required by a half-duplex node to switch from the receive mode to the transmit mode, and the maximum time synchronization error among the nodes and the sink [7]. This ensures that the nodes whose timers expire $\Delta$ after the timer of the best node do not transmit and collide with the contention packet of the best node. Further, $F$ is the sum of the contention packet[2] and feedback packet durations, maximum propagation delay between the nodes and the sink, the switching time, and the maximum time synchronization error among the nodes and the sink. It also accounts for protocol-specific overheads such as mandatory silence durations between two successive transmissions on the channel [26]. In this case, $F/\Delta$ exceeds 1 and can be large.
- *Presence of hidden nodes or when the nodes cannot do carrier-sensing:* In such a system, $\Delta$ is the sum of the maximum propagation delay between the nodes and the sink, the contention packet duration, and the maximum time synchronization error.[3] This ensures that the contention packet of the best node is completely received by the sink before it begins to receive contention packets sent by nodes whose timers expire $\Delta$ later. The value of $F$ is the same as in the previous category, except that now it does not account for the contention packet duration.[4] In this case, $F/\Delta$ is smaller than in the previous category, but it can exceed unity because of protocol-specific overheads [14].

*1) Single-Stage Optimal Timer Mapping:* An optimal metric-to-timer mapping that maximizes the probability of selecting the best node was derived in [14]. In it, the timer of a node expires only at $0, \Delta, 2\Delta, \ldots, N\Delta$, where $N = \lfloor \frac{T_{\max}}{\Delta} \rfloor$, or not at all. When the metric $\mu$ lies in the interval $[1-\alpha_{N,K}[0], 1)$, then the timer expires immediately. When $\mu$ lies in the interval $[1-\alpha_{N,K}[0]-\alpha_{N,K}[1], 1-\alpha_{N,K}[0])$, the timer expires at time $\Delta$. In general, for $i = 0, 1, \ldots, N$, when $\mu$ lies in the interval $\left[1-\sum_{j=0}^{i}\alpha_{N,K}[j], 1-\sum_{j=0}^{i-1}\alpha_{N,K}[j]\right)$, the timer expires at $i\Delta$. Timers of nodes whose metrics lie in the interval $\left[0, 1-\sum_{j=0}^{N}\alpha_{N,K}[j]\right)$ do not expire at all.

It can be seen that the optimal mapping resembles a staircase, in which all the stairs have the same height of $\Delta$, but have different lengths. The mapping is shown in Figure 1. As mentioned, we shall refer to $N$ as the *number of timer levels*. We shall refer to $\alpha_{N,K}[i]$ as the $i$th *stair length* and the interval $(i\Delta, (i+1)\Delta]$ as the $i$th slot.

Specifying the stair lengths $\alpha_{N,K}[i]$, for $i = 0, \ldots, N$, completely specifies the optimal timer mapping. The optimal stair lengths that maximize the probability of success $P_{N,K}$ are given by the following recursion [14]:

$$\alpha_{N,K}[j] = \begin{cases} \frac{1-P_{N-1,K}}{K-P_{N-1,K}}, & j=0 \\ (1-\alpha_{N,K}[0])\alpha_{N-1,K}[j-1], & 1 \le j \le N \end{cases}, \tag{1}$$

where $\alpha_{0,K}[0] = \frac{1}{K}$. The success probability $P_{N,K}$ is given in terms of the stair lengths as

$$P_{N,K} = K \sum_{i=0}^{N} \alpha_{N,K}[i] \left(1 - \sum_{j=0}^{i} \alpha_{N,K}[j]\right)^{K-1}. \tag{2}$$

[1]However, in a cooperative decode-and-forward relaying scenario, where a source node communicates to a destination node with the help of intermediate relay nodes, the number of relays that successfully decode the transmission by the source is a random variable, and depends on the instantaneous realizations of the source-to-relay channels. In such a case, the algorithm needs to be designed keeping in mind the total number of relays and the statistics of the number of relays that can decode the transmission by source, as has been done in [20].

[2]Note that the contention packet duration is included in $F$ because the sink will transmit its feedback message only after it has received the contention packet transmitted by the node. In the case under consideration, the node can start transmitting its contention packet as late as $N\Delta$.

[3]In this case, $\Delta$ does not include the switching time duration because the transmission of the contention packet by all the nodes is delayed by the switching time duration.

[4]In [7], a flag frame is broadcast in every slot by the sink in order to handle hidden nodes. However, this increases the size of $\Delta$. In our scheme, a flag frame is not required since feedback is sent at the end of the timer stage.

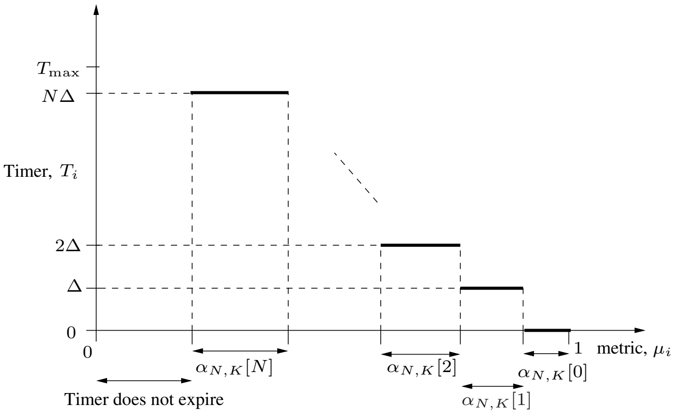


Fig. 1. Illustration of the optimal metric-to-timer mapping when the selection duration is $T_{\max}$. Here, $N = \left\lfloor \frac{T_{\max}}{\Delta} \right\rfloor$.

Note that the optimal stair lengths are unequal, unlike [17]. From the above recursion, we see that stair lengths are a function of the number of timer levels and the number of nodes. Both these parameters need to be specified in order to completely specify the single-stage optimal timer mapping.

### B. Properties of the Single-Stage Optimal Timer Scheme

In Appendix A, we derive the following new results about the single-stage optimal timer scheme with $N$ timer levels designed for $K$ nodes. These results help in the design of the proposed scheme in the next section.

1) The success probability is given by
$$P_{N,K} = \left( \frac{K-1}{K - P_{N-1,K}} \right)^{K-1}, \tag{3}$$
for $N \geq 0$ and $K \geq 2$, with $P_{-1,K}$ defined to be 0. Further, for $K = 2$ nodes, the success probability expression simplifies to
$$P_{N,2} = \frac{N+1}{N+2}. \tag{4}$$
2) The success probability $P_{N,K}$ strictly increases as $N$ increases and strictly decreases as $K$ increases. Thus, increasing the number of timer levels increases the success probability, while increasing the number of nodes decreases the success probability.
3) We say that a collision occurs in slot $s$ when the timers of the best node and at least one other node expire at time $s\Delta$. Let $Y_{N,K}^{(s)}$ denote the number of nodes that collided in slot $s$ given that a collision has occurred in the slot $s$ and no transmission in any slot before the slot $s$. Then, the distributions of $Y_{N,K}^{(s)}$ and $Y_{N-s,K}^{(0)}$ are the same for all $s = 0, \ldots, N$. Thus, their expected values are also equal.
4) The average value of $Y_{N,K}^{(s)}$ lies between 2 and 2.4 for any $N \geq 0$, $K \geq 2$, and $s = 0, \ldots, N$.

## III. Proposed Multi-Stage Timer Scheme

When a single-stage optimal timer designed for $N$ timer levels and $K$ nodes is run, one of the following three outcomes occurs:

- *Success:* This occurs when the timer of the best node expires and the timer of the second best node does not expire in the same time slot as the best node's timer.
- *Collision in slot* $s$: This occurs when the timers of the best node and the second best node expire at $s\Delta$, i.e., a collision occurs in slot $s$.
- *Idle*: This occurs when the metrics of all the nodes lie in the interval $\left[0, 1 - \sum_{j=0}^{N} \alpha_{N,K}[j]\right)$.

The key idea of the proposed scheme is to run the single-stage timer scheme multiple times depending on the outcomes. A timer stage with $N$ timer levels is run for a duration of $(N+1)\Delta$. This is followed by a broadcast of its outcome by the sink, which is a success, idle, or collision, which takes a duration of $F$. In case of a collision, the sink also broadcasts the slot number in which the collision occurred. This process is continued until a success occurs.

The main design question that arises is how many timer levels to choose for each stage and for how many nodes to design each stage for. In order to gain insights into this design problem, we first consider the special case of $K = 2$ nodes. Thereafter, we consider the general case with $K > 2$ nodes.

### A. System with Two Nodes ($K = 2$)

A priori, we know that the metrics of the two nodes are uniformly distributed in the interval $[0, 1)$. Let an $N$-level single-stage timer designed for $K = 2$ nodes be run. The idle and collision outcomes provide useful information about where the metric of the best node lies. This is brought out by the following two observations.

*Observation 1:* A collision outcome in slot $s$ implies that both the nodes must have transmitted in that slot. Therefore, conditioned on this event, the metrics of both the nodes must be uniformly distributed in the interval $\left[1 - \sum_{j=0}^{s} \alpha_{N,2}[j], 1 - \sum_{j=0}^{s-1} \alpha_{N,2}[j]\right)$. Therefore, the following RV
$$\nu_i = \frac{\mu_i - 1 + \sum_{j=0}^{s} \alpha_{N,2}[j]}{\alpha_{N,2}[s]}, \tag{5}$$
is uniformly distributed over the interval $[0, 1)$ for both nodes, namely $i = 1, 2$.

*Observation 2:* An idle outcome implies that the timer of neither node expired. Therefore, conditioned on this event, the metrics of both the nodes must lie in the interval $\left[0, 1 - \sum_{j=0}^{N} \alpha_{N,2}[j]\right)$. Therefore, the following RV
$$\nu_i = \frac{\mu_i}{1 - \sum_{j=0}^{N} \alpha_{N,2}[j]}, \tag{6}$$
is uniformly distributed over the interval $[0, 1)$ for both nodes, namely $i = 1, 2$.

The above two observations show how the metrics should be rescaled in the event of a collision or idle, so that a new timer stage can be run. Notice that when a collision or an idle slot occurs, the number of nodes that will participate in the next time stage is always two. Therefore, all the timer stages are designed with the same number of timer levels $N$ and for two nodes. Notice also that the rescaling of $\mu_i$ to $\nu_i$ preserves order, i.e., $\mu_1 > \mu_2$ if and only if $\nu_1 > \nu_2$.

*1) Specification of Proposed Scheme:* The proposed scheme is as follows:

*Initialization:* Given $K = 2$ and $N$, evaluate $\alpha_{N,2}[i]$, for $0 \leq i \leq N$. A node $i$ sets its metric $\nu_i$ as $\nu_i \leftarrow \mu_i$.

*Transmission rule:* A node $i$ whose metric lies in the interval $\left[1 - \sum_{j=0}^{s} \alpha_{N,2}[j]\,, 1 - \sum_{j=0}^{s-1} \alpha_{N,2}[j]\right)$ transmits at $s\Delta$.

*Feedback from sink:* After a time of $(N+1)\Delta$, after which no timer expires, the sink feeds back an outcome.[5] The sink feeds back an idle outcome, if no transmission occurred, a success outcome if one node transmitted and it, thus, could decode the transmission, or a collision outcome, if the first slot in which a transmission occurred resulted in a collision. The sink also feeds back the index of the slot in which the collision occurred.

*On success outcome:* Then the scheme terminates.

*On idle outcome:* Then each node $i$ rescales its metric to $\nu_i \leftarrow \frac{\nu_i}{1-\sum_{j=0}^{N} \alpha_{N,2}[j]}$ and the $N$-level timer scheme is rerun.

*On collision in slot s outcome:* Then each node $i$ rescales its metric to $\nu_i \leftarrow \frac{\nu_i - 1 + \sum_{j=0}^{s} \alpha_{N,2}[j]}{\alpha_{N,2}[s]}$ and the $N$-level timer scheme is rerun.

*Comments:* Since the rescalings in (5) and (6) preserve order, it is guaranteed that the first successful transmission to the sink is from the best node. The pseudo-code for the scheme is given in Figure 2.

**Data**: $K = 2$, $N$, $\mu_1$, and $\mu_2$
Evaluate $\alpha_{N,2}[i]$, for $0 \leq i \leq N$;
**begin**
  $\nu_i \leftarrow \mu_i$, for $i = 1, 2$; *outcome* $\leftarrow$ NIL;
  **while** *outcome* $\neq$ *SUCCESS* **do**
    Run $N$-level timer scheme designed for 2 nodes;
    Metric of node $i$ is $\nu_i$;
    Sink feeds back *outcome*;
    **if** *outcome* = *COLLISION in slot* $s$ **then**
      $\nu_i \leftarrow \frac{\nu_i - 1 + \sum_{j=0}^{s} \alpha_{N,2}[j]}{\alpha_{N,2}[s]}$;
    **end**
    **else if** *outcome* = *IDLE* **then**
      $\nu_i \leftarrow \frac{\nu_i}{1-\sum_{j=0}^{N} \alpha_{N,2}[j]}$;
    **end**
  **end**
**end**

Fig. 2. Proposed scheme for a system with 2 nodes.

*2) Analysis:* We now derive an expression for the average time required to select the best node and the number of timer levels $N^*$ that minimizes it.

**Theorem** *1:* For a system with two nodes, the average time $T_2$ required to select the best node is given by

$$T_2 = (N+2)\Delta + \left(\frac{N+2}{N+1}\right) F. \tag{7}$$

For $F > 0$, $N^* = \sqrt{\frac{F}{\Delta}} - 1$, if $\sqrt{\frac{F}{\Delta}} - 1$ is an integer, else, it is either $\left\lfloor \sqrt{\frac{F}{\Delta}} - 1 \right\rfloor$ or $\left\lceil \sqrt{\frac{F}{\Delta}} - 1 \right\rceil$. For $F = 0$, $N^* = 0$.

*Proof:* The proof is relegated to Appendix B. ■

[5]In the no hidden node case, the sink will wait for the transmission by the nodes to end and only then commence transmission of its feedback packet.

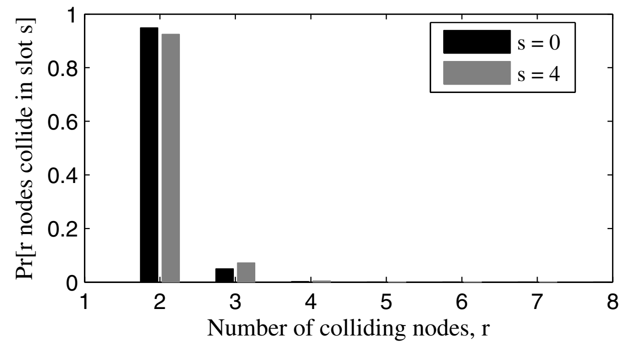


Fig. 3. Probability mass function of the number of nodes that collided at $s\Delta$ given that a collision occurred in slot $s$ ($N = 10$ and $K = 100$).

Theorem 1 shows that the optimal number of levels that a timer stage should use is proportional to $\sqrt{F/\Delta}$. Therefore, as $F$ increases, the number of levels in every timer stage increases. From the properties of the single-stage optimal timer scheme in Section II-B, this implies that the probability of success of a single timer stage increases. Thus, as the feedback overhead increases, the proposed scheme chooses a larger number of levels in each timer stage. By doing so, it reduces the odds of subsequently using more timer stages and, consequently, more feedback.

### *B. General System with $K > 2$ Nodes*

As in the two nodes case, it can be seen that, in case of an idle outcome, the following rescaling of metrics from $\mu_i$ to $\nu_i$ ensures that $\nu_i \sim U[0,1)$ for all the nodes:

$$\nu_i = \frac{\mu_i}{1 - \sum_{j=0}^{N} \alpha_{N,K}[j]}. \tag{8}$$

Similarly, in case of a collision outcome in slot $s$, the following rescaling ensures that $\nu_i \sim U[0,1)$ for the nodes that collided, one of which is the best node:

$$\nu_i = \frac{\mu_i - 1 + \sum_{j=0}^{s} \alpha_{N,K}[j]}{\alpha_{N,K}[s]}. \tag{9}$$

As before, the above rescalings also preserve order.

As in the $K = 2$ nodes case, in the event of an idle outcome, all the nodes that participated in the current timer stage should participate again in the next timer stage. However, in case of a collision outcome, the number of nodes that collided is a random variable and is not known. It can take any value between 2 and $K$. Therefore, for this outcome, it is not obvious how many nodes to design the next timer stage for. Figure 3 plots the probability mass function of the number of nodes that collided given that a collision has occurred in slot $s$, for $N = 10$ and $K = 100$. We observe for all the slots that the probability that two nodes collided is much larger than the probability that three or more nodes collided. *Therefore, once a collision occurs, one can design all the subsequent timer stages for two nodes.*

We note that such an intuition has also been used in the design of the splitting scheme [12] and the first come first serve (FCFS) multiple access protocol [23]. Another result that supports the above conclusion is Theorem 2, which is stated in Appendix A. It shows that the average number of nodes that collided, given that a collision has occurred in slot $s$, lies in a narrow range between 2 and 2.4 for any $N \geq 0$, $K \geq 2$, and $s \in \{0, 1, \ldots, N\}$. From the Markov inequality [21], it then follows that the probability that at least $l$ nodes collide is less than $0.4/(l-2)$, for $l \geq 3$. This rules out the subtle possibility that the rest of the probability mass is concentrated on an event that involves a collision among $q$ nodes, where $q$ grows with $K$.

*1) Proposed Scheme for $K > 2$ Nodes:* In summary, before the first collision outcome occurs, each timer stage should be designed for $K$ nodes. However, once a collision occurs, the number of nodes the timer stage should be designed for drops to two. Therefore, for $K > 2$, the proposed scheme starts with a timer stage designed for $K$ nodes with $N_I$ timer levels. It persists with this timer stage until a collision occurs. Thereafter, in case of a collision, it switches to a timer stage designed for two nodes. Since the number of nodes has changed, the number of timer levels is also changed to $N_C$. The two parameters $N_I$ and $N_C$ will be optimized later on.

The scheme is specified as follows:

*Initialization:* Given $K$, $N_I$, and $N_C$, evaluate $\alpha_{N_I,K}[i]$, for $0 \leq i \leq N_I$, and $\alpha_{N_C,2}[i]$, for $0 \leq i \leq N_C$. A node $i$ sets its metric $\nu_i$ as $\nu_i \leftarrow \mu_i$.

*Transmission rule:* Run the $N_I$-level single-stage timer scheme designed for $K$ nodes if no collision has occurred previously or if it is the first timer stage. If, on the other hand, a collision has occurred previously, then run the $N_C$-level single-stage timer scheme designed for 2 nodes.

*Feedback:* The sink feeds back an idle outcome, if no transmission occurred, a success outcome if one node transmitted, and it, thus, could decode the transmission, or a collision outcome otherwise. In case of a collision outcome, the sink also feeds back the index of the slot in which the collision occurred.

*On success outcome:* The scheme terminates.

*On collision in slot $s$ outcome:* Each node $i$ that was involved in the collision rescales its metric to the new metric $\nu_i \leftarrow \frac{\nu_i - 1 + \sum_{j=0}^{s} \alpha_{N_I,K}[j]}{\alpha_{N_I,K}[s]}$, if no collision has occurred before, and to $\nu_i \leftarrow \frac{\nu_i - 1 + \sum_{j=0}^{s} \alpha_{N_C,2}[j]}{\alpha_{N_C,2}[s]}$, if a collision has occurred before. All nodes not involved in the collision do not participate in any of the subsequent timer stages.[6]

*On idle outcome:* If a collision had occurred before, then each node $i$ that participated in the previous timer stage, rescales its metric value to $\nu_i \leftarrow \frac{\nu_i}{1 - \sum_{j=0}^{N_C} \alpha_{N_C,2}[j]}$ and the $N_C$-level timer is run. On the other hand, if no collision had occurred in any of the previous timer stages then each node $i$ rescales its metric value to $\nu_i \leftarrow \frac{\nu_i}{1 - \sum_{j=0}^{N_I} \alpha_{N_I,K}[j]}$ and the $N_I$-level timer scheme is run.

The pseudo-code of this scheme is given in Figure 4.

As shown in Appendix C, the average selection time $T_K$ of the proposed scheme is given by

**Data**: $K$, $\mu_1, \ldots, \mu_K$, $N_I$, and $N_C$
Evaluate $\alpha_{N_I,K}[i]$, for $0 \leq i \leq N_I$, and $\alpha_{N_C,2}[i]$, for $0 \leq i \leq N_C$;
**begin**
  $\nu_i \leftarrow \mu_i, \forall i = 1$ to $K$;  *outcome* $\leftarrow$ NIL;
  and *Collision-occurred* $\leftarrow$ NO;
  **while** *outcome* $\neq$ *SUCCESS* **do**
    **if** *Collision-occurred = NO* **then**
      Run $N_I$-level timer designed for $K$ nodes;
      Metric of node $i$ is $\nu_i$;
      Sink feeds back *outcome*;
      **if** *outcome = COLLISION in slot* $s$ **then**
        $\nu_i \leftarrow \frac{\nu_i - 1 + \sum_{j=0}^{s} \alpha_{N_I,K}[j]}{\alpha_{N_I,K}[s]}$ and
        *Collision-occurred* $\leftarrow$ YES;
      **end**
      **else if** *outcome = IDLE* **then**
        $\nu_i \leftarrow \frac{\nu_i}{1 - \sum_{j=0}^{N_I} \alpha_{N_I,K}[j]}$;
      **end**
    **end**
    **else**
      % Collision-occurred = YES
      Run $N_C$-level timer designed for 2 nodes;
      Metric of node $i$ is $\nu_i$;
      Sink feeds back *outcome*;
      **if** *outcome = COLLISION in slot* $s$ **then**
        $\nu_i \leftarrow \frac{\nu_i - 1 + \sum_{j=0}^{s} \alpha_{N_C,2}[j]}{\alpha_{N_C,2}[s]}$;
      **end**
      **else if** *outcome = IDLE* **then**
        $\nu_i \leftarrow \frac{\nu_i}{1 - \sum_{j=0}^{N_C} \alpha_{N_C,2}[j]}$;
      **end**
    **end**
  **end**
**end**

Fig. 4. Proposed scheme for a system with $K$ nodes.

$$T_K \approx \frac{(N_I+1)\Delta + F}{1 - \left(1 - \sum_{j=0}^{N_I} \alpha_{N_I,K}[j]\right)^K} + \frac{(N_C+2)\left(\Delta + \frac{F}{N_C+1}\right)}{1 - \left(1 - \sum_{j=0}^{N_I} \alpha_{N_I,K}[j]\right)^K} \times \sum_{r=2}^{K} \sum_{j=0}^{N_I} \binom{K}{r} \alpha_{N_I,K}[j]^r \left(1 - \sum_{l=0}^{j} \alpha_{N_I,K}[l]\right)^{K-r}. \quad (10)$$

The value of $N_C$, denoted by $N_C^*$, that minimizes $T_K$ is as follows.

**Proposition** *1:* For $F > 0$, $N_C^* = \sqrt{\frac{F}{\Delta}} - 1$, if $\sqrt{\frac{F}{\Delta}} - 1$ is an integer, else it is either $\left\lfloor \sqrt{\frac{F}{\Delta}} - 1 \right\rfloor$ or $\left\lceil \sqrt{\frac{F}{\Delta}} - 1 \right\rceil$. For $F = 0$, $N_C^* = 0$.

*Proof:* The proof is relegated to Appendix D. ■

Notice that $N_C^*$ does not depend on $K$. We have, thus, reduced the design of the optimal multi-stage timer scheme to an optimization of a single parameter $N_I$. This can be

[6] A node can determine if it was involved in a collision in slot $s$ or not from the feedback broadcast by the sink.

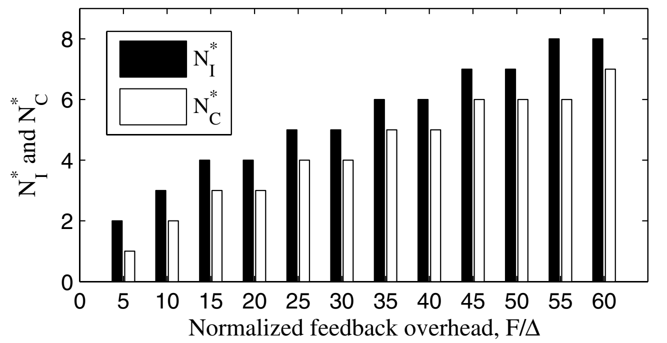


Fig. 5. $N_I^*$ and $N_C^*$ as a function of normalized feedback overhead, $\frac{F}{\Delta}$, for the proposed scheme ($K = 5$).

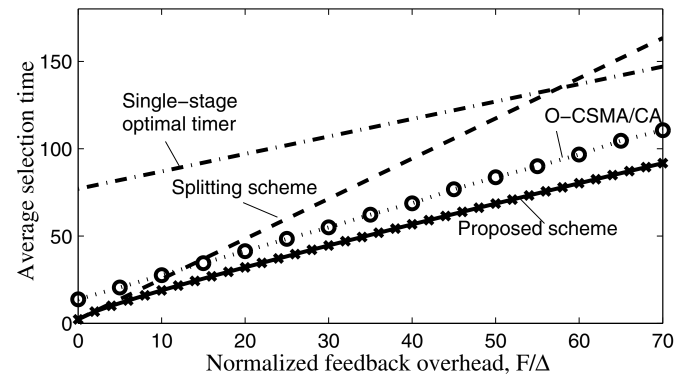


Fig. 6. Comparison of the average selection time vs. normalized feedback overhead for various schemes ($K = 5$). Markers (x) indicate simulation results for the proposed scheme, where as the solid line (–) indicates analysis using (10).

implemented efficiently using binary search, for example. Figure 5 plots $N_I^*$, which denotes the optimal $N_I$, and $N_C^*$ as a function of $\frac{F}{\Delta}$. We see that both $N_I^*$ and $N_C^*$ increase as $\frac{F}{\Delta}$ increases. This is because, as the feedback overhead increases, the number of levels in each timer stage increases so as to increase the success probability of each stage. This reduces the need for subsequent timer stages and feedback. We also observe that $N_I^* \geq N_C^*$. This is because the average number of nodes that participate after a collision decreases from $K$ to approximately 2.

## IV. Numerical Results and Performance Implications

We now evaluate the performance of the proposed scheme using Monte Carlo simulations that use $30,000$ samples, and compare it with several schemes in the literature, which are briefly described below.

1) *O-CSMA/CA:* In O-CSMA/CA, a timer mapping with $M$ levels and equal stair lengths is used in the first stage. In case of a failure, the number of levels is doubled, until it reaches $M_{\max}$. The collided nodes participate in the next stage with probability $\frac{1}{2}$. Therefore, O-CSMA/CA may not select the best node. We use $M = 7$ and $M_{\max} = 1023$, as per [17], [26].

2) *Polling:* Each node is scheduled to transmit a contention packet of duration $P$, which contains its metric value, in a pre-assigned slot. Subsequently, the sink feeds back a message of duration $F$ to indicate which node has been selected. The selection scheme requires a time $KP+F$ to select. In both the scenarios with and without hidden nodes, $P \geq \Delta$. Therefore, we conservatively set the selection duration of the polling scheme to be $K\Delta + F$.

3) *Threshold-based random access [18]:* This scheme allocates $L$ slots for contention, which consumes a time of $L\Delta + F$. Note that this is again a conservative estimate. Here, $F$ accounts for the feedback at the end of the $L$ slots. A node $i$ whose metric exceeds a threshold $\eta$ transmits in each of the $L$ slots independently with probability $q$. In each slot an idle, success, or collision outcome can occur. Among the success slots (if any), the sink chooses one with uniform probability and notifies the node that transmitted in that slot. In case there is no success slot, then we assume that the selection fails and a time duration of $L\Delta + F$ is wasted.

4) *Random selection:* In this scheme, the sink chooses one out of the $K$ nodes with uniform probability. Each time the sink does this, it takes a time of $F$.

### A. Selection Speed and Scalability

Figure 6 compares the average selection time of the proposed scheme with that of the splitting scheme as a function of the normalized feedback overhead $\frac{F}{\Delta}$ for $K = 5$ nodes. Also shown is the average time required to select a node by O-CSMA/CA and the single-stage optimal timer. Since the single-stage timer scheme cannot guarantee success, its parameters are chosen to ensure a high success probability of $98\%$. The success probability of O-CSMA/CA turns out to be $81\%$. We observe that the proposed scheme is faster than all the benchmark schemes and, yet, ensures a $100\%$ success probability. For example, for $\frac{F}{\Delta} = 20$ and $K = 5$, splitting, O-CSMA/CA, and single-stage optimal timer take $50\%$, $28\%$, and $200\%$ more time, respectively, than the proposed scheme to select. Moreover, the relative gains increase as $\frac{F}{\Delta}$ increases. Note that when $F/\Delta \leq 1$, the proposed scheme reduces to the splitting scheme of [13] in the sense that both divide the metric interval in which a collision has occurred into two parts. Consequently, the curves of the average selection times of the proposed scheme and the splitting scheme merge.

Figure 7 plots the average time to select as a function of $K$ for $\frac{F}{\Delta} = 20$. We see that the proposed scheme scales well with $K$. Its average selection time increases by just $4\%$ when $K$ increases from 5 to 100, which is the least increase among all the schemes. Moreover, as $K$ increases, the performance gap between the proposed scheme and the benchmark schemes increases. For example, for $K = 100$, splitting, O-CSMA/CA, and single-stage optimal timer take $54\%$, $536\%$, and $244\%$ more time, respectively. Figures 6 and 7 also verify that the expression in (10) for the average time required by the proposed scheme to select is accurate.

Figure 8 plots the average number of feedback messages required by the proposed scheme and compares it with O-CSMA/CA and splitting. We see that far fewer feedback messages are required by the proposed scheme. For example, for $\frac{F}{\Delta} = 20$ and $K = 50$, splitting and O-CSMA/CA

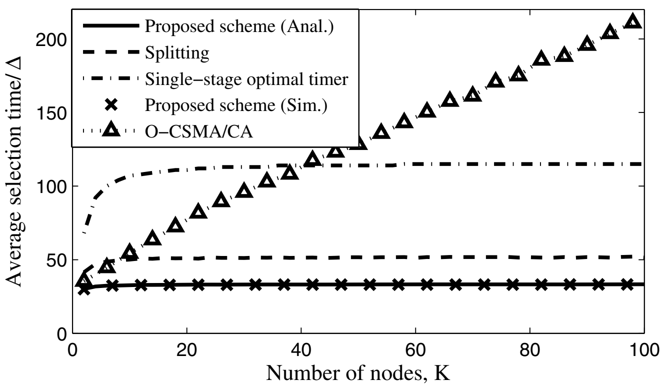


Fig. 7. Comparison of the average selection time vs. number of nodes ($\frac{F}{\Delta} = 20$).

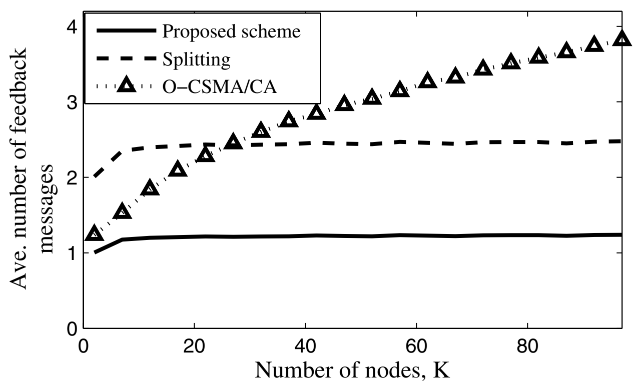


Fig. 8. Comparison of the average number of feedback messages required during the selection phase vs. number of nodes ($\frac{F}{\Delta} = 20$).

require $100\%$ and $149\%$, respectively, more feedback than the proposed scheme.

### B. Net Throughput Implications

The benchmark schemes mentioned above compromise on the goal of selecting the best node in different ways. Some of them may not select the best node in order to quickly wrap up the selection process. However, this comes at the expense of a lower data rate when the selected node is subsequently used for data transmission. Therefore, to demonstrate the relevance of the proposed scheme and to compare the diverse approaches, we compare the net throughput achieved by all these schemes. The net throughput takes into account the time overheads of selection and the throughput penalties associated with not choosing the best node.

The system that we consider consists of $K$ nodes and a BS, which also acts as the sink. The metric of a node $i$ is the channel power gain $h_i$ of the link from the BS to the node. Here, $h_i$ is an exponential RV with unit mean, which models Rayleigh fading. The selected node adapts its rate as a function of its channel gain to transmit a data packet of $Q$ bits. For the sake of illustration, we use the Shannon capacity formula to determine the rate as a function of channel gain: $W \log_2 (1 + \omega \rho h_S)$, where $S$ is the selected node, $\rho$ is the transmit signal-to-noise ratio (SNR), $W$ is the channel bandwidth, and $\omega$ is the coding loss of the channel code [2,

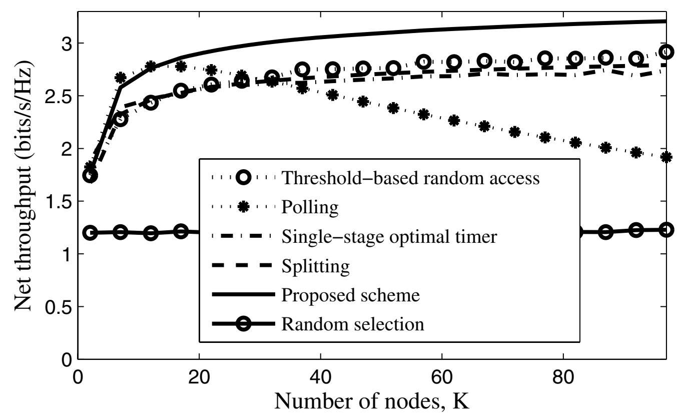


Fig. 9. Net throughput as a function of $K$ for various schemes ($Q = 4000$ bits, $\frac{F}{\Delta} = 20$, and $\Delta = \frac{10}{W}$).

Chap. 9]. We assume that the feedback and the contention packets are transmitted at a constant rate independent of the channel conditions. The channel is assumed to remain constant for the selection and transmission durations.

Figure 9 compares the net throughput in bits/s/Hz as a function of $K$ of the proposed scheme and the benchmark schemes, for $Q = 4000$ bits, $\frac{F}{\Delta} = 20$, $\omega = 1$, and $\Delta = \frac{10}{W}$ (i.e., 10 symbol durations).[7] For the parameters considered, we use $L = 6$ slots for threshold-based random access and 8 levels for the single-stage optimal timer, as they maximize their respective throughputs.[8] We observe that the net throughput of the proposed scheme increases with the number of nodes, which shows its ability to exploit multiuser diversity. At $K = 100$, the throughput gains over splitting, single-stage optimal timer, threshold-based random access, polling, and random selection are $15\%$, $17\%$, $10\%$, $67\%$, and $145\%$, respectively.

Figure 10 compares the net throughput of various schemes as a function of the feedback overhead, $\frac{F}{\Delta}$, for $Q = 20000$ bits, $K = 100$, and $\Delta = \frac{10}{W}$. We see that the proposed scheme outperforms all other schemes. As $\frac{F}{\Delta}$ increases, the throughput of the splitting scheme decreases much faster than the proposed scheme. Notice that polling outperforms both threshold-based random access and the single-stage optimal timer. This again reinforces the need to invest time to select the best node and exploit multi-user diversity gains.

## V. Conclusions

We proposed a novel distributed selection scheme that inherits the best features of the timer and splitting schemes, generalizes both of them, and guarantees best node selection. The scheme runs in stages with the single-stage timer scheme

[7] In random selection, the average transmission duration for a given payload of $Q$ bits is unbounded for Rayleigh fading. Therefore, to address this issue, the chosen node is allowed to transmit only if its channel gain exceeds a threshold. Otherwise, another node is chosen, again uniformly, from the remaining nodes. The threshold is chosen such that the channel gain of each node exceeds it with a probability of 0.95.

[8] The parameters $\eta$, $q$, and $L$ of the threshold-based random access scheme are numerically optimized as a function of $K$, $Q$, and the CDF of the channel power gain in order to maximize its net throughput and to provide as fair a comparison as possible.

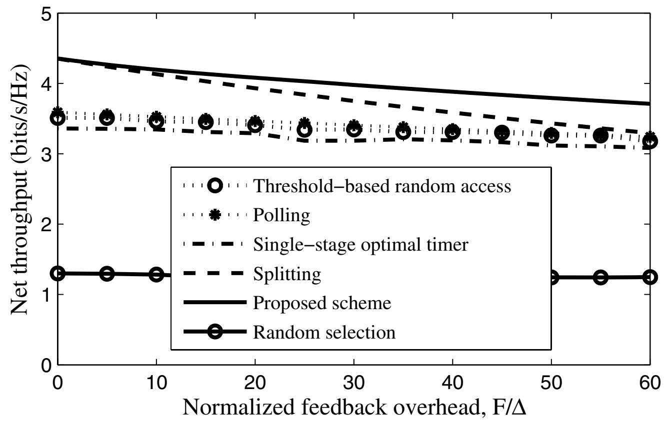


Fig. 10. Net throughput as a function of $\frac{F}{\Delta}$ of various schemes ($Q = 20000$ bits, $K = 100$, and $\Delta = \frac{10}{W}$).

used in every stage. The number of timer levels is either $N_I^*$ or $N_C^*$, and depends on whether a collision has occurred or not in the past. As the feedback overhead increases, we saw that these levels increase so as to increase the probability of success in each stage and reduce the need for subsequent feedback. The proposed scheme reduces to the splitting scheme only when the feedback duration is smaller than the vulnerability window.

We saw that the proposed scheme is much faster and requires far fewer feedback messages than the splitting scheme and O-CSMA/CA, which doubles the contention window duration in case of a collision. Interestingly, in our scheme, the timer stage duration shrinks once a collision occurs. We saw that the proposed scheme scales well with the number of nodes in the system, unlike the polling scheme. We also saw that feedback-aware fast, reliable, and scalable selection translates into a significant net throughput gain. The proposed scheme can be speeded up further by allowing the sink to transmit its feedback message as soon as it receives the first contention packet instead of waiting for the entire duration of the timer stage.

## Appendix

### A. *Properties of the Single-Stage Optimal Timer Scheme*

We first present a recursive expression for the success probability of the optimal timer scheme.

**Proposition** *2:* The probability of success satisfies the following recursion:

$$P_{N,K} = \left(\frac{K-1}{K - P_{N-1,K}}\right)^{K-1}, \quad \text{for} \quad N \geq 0, \ K \geq 2, \quad (11)$$

where $P_{-1,K} \triangleq 0$, for all $K \geq 1$. When $K = 2$, it simplifies to

$$P_{N,2} = \frac{N+1}{N+2}, \quad \text{for} \quad N \geq 0. \quad (12)$$

*Proof:* 1) Due to space constraints, we directly start with (20) of [14], which relates the success probability of the single-stage optimal timer with $N$ levels to that with $(N-1)$ levels:

$$P_{N,K} = K\alpha_{N,K}[0]\,(1-\alpha_{N,K}[0])^{K-1} + (1-\alpha_{N,K}[0])^{K} P_{N-1,K}. \quad (13)$$

Substituting $\alpha_{N,K}[0]$ from (1) into (13) yields (11).

2) For $K = 2$, (11) reduces to $P_{N,2} = \frac{1}{2 - P_{N-1,2}}$. Using induction on $N$, for $N \geq -1$, we can show that $P_{N,2} = \frac{N+1}{N+2}$. The steps are omitted to conserve space. ■

We see from (12) that $P_{N,2}$ is an increasing function of $N$. The following result generalizes this to $K \geq 2$. It proves that $P_{N,K}$ strictly increases as $N$ increases and it strictly decreases as $K$ increases. This result was previously shown in [14] only for the asymptotic case of a large number of nodes.

**Proposition** *3:* For all $N \geq 0$ and $K \geq 2$,

1) $P_{N,K}$ is a strictly increasing function in $N$, i.e., $P_{N,K} > P_{N-1,K}$.
2) $P_{N,K}$ is a strictly decreasing function in $K$, i.e., $P_{N,K} > P_{N,K+1}$.

*Proof:* We first prove the following lemma, which will be useful in the proof.

**Lemma** *1:* For any $r \in (0,1)$ and an integer $p \geq 2$, we have $r < \left(\frac{p-1}{p-r}\right)^{p-1}$.

*Proof:* Consider a function $g_r(x) = \left(\frac{x-1}{x-r}\right)^{x-1}$, for $x \in \mathbb{R}$, $x \geq 2$, and $r \in (0,1)$. Then, by using the first order condition and the inequality $\log(a) < a - 1$, for $a \in (0,1)$, it can be shown that $g_r(x)$ is a strictly decreasing function in $x$. Hence, $\left(\frac{p-1}{p-r}\right)^{p-1} > \lim_{m\to\infty}\left(\frac{m-1}{m-r}\right)^{m-1} = e^{r-1} > r$. ■

1) From Proposition 2, we have $P_{N,K} = \left(\frac{K-1}{K-P_{N-1,K}}\right)^{K-1}$. Since $P_{N,K} \in (0,1)$ and $K \geq 2$, it follows from Lemma 1 that $P_{N,K} > P_{N-1,K}$.

2) We prove $P_{N,K+1} < P_{N,K}$ using mathematical induction over $N$, for $N \geq 0$.

(i) $N = 0$ *case:* From Proposition 2 and $P_{-1,K} = 0$, we get $P_{0,K} = \left(\frac{K-1}{K}\right)^{K-1}$. It can be shown using the first order condition and the inequality $\log(a) < a - 1$, for $a \in [\frac{1}{2}, 1)$, that the function $g(x) = \left(\frac{x-1}{x}\right)^{x-1}$, for $x \in \mathbb{R}$ and $x \geq 2$, is a strictly decreasing function in $x$. Hence, $\left(\frac{K-1}{K}\right)^{K-1} > \left(\frac{K}{K+1}\right)^{K}$. Therefore, the result follows.

(ii) Let $P_{M,K+1} < P_{M,K}$ be true for some $M \geq 0$. From Proposition 2, we have

$$P_{M,K} = K - \frac{K-1}{(P_{M+1,K})^{\frac{1}{K-1}}}, \quad (14)$$

and

$$P_{M,K+1} = K + 1 - \frac{K}{(P_{M+1,K+1})^{\frac{1}{K}}}. \quad (15)$$

Thus, the hypothesis $P_{M,K+1} < P_{M,K}$ implies

$$K + 1 - \frac{K}{(P_{M+1,K+1})^{\frac{1}{K}}} < K - \frac{K-1}{(P_{M+1,K})^{\frac{1}{K-1}}}. \quad (16)$$

This can be written as $\frac{K-1}{(P_{M+1,K})^{\frac{1}{K-1}}} < \frac{K-(P_{M+1,K+1})^{\frac{1}{K}}}{(P_{M+1,K+1})^{\frac{1}{K}}}$. Raising both sides of the inequality to the power $K-1$, we get

$$\frac{(K-1)^{K-1}}{P_{M+1,K}} < \frac{\left(K - (P_{M+1,K+1})^{\frac{1}{K}}\right)^{K-1}}{(P_{M+1,K+1})^{\frac{K-1}{K}}}. \quad (17)$$

This simplifies to

$$\left(\frac{K-1}{K-(P_{M+1,K+1})^{\frac{1}{K}}}\right)^{K-1} < \frac{P_{M+1,K}}{(P_{M+1,K+1})^{\frac{K-1}{K}}}. \quad (18)$$

From Lemma 1, we also know that

$$\left(\frac{K-1}{K-(P_{M+1,K+1})^{\frac{1}{K}}}\right)^{K-1} > (P_{M+1,K+1})^{\frac{1}{K}}. \quad (19)$$

Substituting this in (18) yields $P_{M+1,K+1} < P_{M+1,K}$. Hence, the result follows. ■

We now show how the stair lengths of the $N$-level single-stage optimal timer, $\alpha_{N,K}[i]$, are related to those of the $(N-r)$-level single-stage optimal timer.

**Proposition** *4:* For $0 \leq r \leq N$ and $r \leq i \leq N$ we have,

$$\alpha_{N,K}[i] = \alpha_{N-r,K}[i-r]\left(1-\sum_{j=0}^{r-1}\alpha_{N,K}[j]\right). \quad (20)$$

*Proof:* Let $N \geq 0$ and $K \geq 2$. For $r=0$ and $0 \leq i \leq N$ the proof is trivial. For $r=1$ and $1 \leq i \leq N$, (20) is the same as (1), and we are done.

Now, let (20) be true for some $1 \leq r < N$ and for all $i$ such that $r \leq i \leq N$. Call this hypothesis $\mathcal{H}_r$. We shall show that $\mathcal{H}_{r+1}$ is also true. From (1), we know that

$$\alpha_{N-r-1,K}[i-r-1] = \frac{\alpha_{N-r,K}[i-r]}{(1-\alpha_{N-r,K}[0])}, \text{ for } r+1 \leq i \leq N. \quad (21)$$

Using hypothesis $\mathcal{H}_r$ for $i=r$ we get

$$1-\alpha_{N-r,K}[0] = \frac{1-\sum_{j=0}^{r}\alpha_{N,K}[j]}{1-\sum_{j=0}^{r-1}\alpha_{N,K}[j]}. \quad (22)$$

Substituting (22) into (21) yields

$$\alpha_{N-r-1,K}[i-r-1] = \alpha_{N-r,K}[i-r]\left(\frac{1-\sum_{j=0}^{r-1}\alpha_{N,K}[j]}{1-\sum_{j=0}^{r}\alpha_{N,K}[j]}\right). \quad (23)$$

Using the induction hypothesis $\mathcal{H}_r$, which states that $\alpha_{N-r,K}[i-r]\left(1-\sum_{j=0}^{r-1}\alpha_{N,K}[j]\right) = \alpha_{N,K}[i]$, we get

$$\alpha_{N-r-1,K}[i-r-1]\left(1-\sum_{j=0}^{r}\alpha_{N,K}[j]\right) = \alpha_{N,K}[i], \quad (24)$$

for $r+1 \leq i \leq N$. Hence, $\mathcal{H}_{r+1}$ is also true. ■

We now delve into the event in which a collision occurs. Let $X_{N,K}^{(s)}$ denote the number of nodes that set their timer values to $s\Delta$, for $0 \leq s \leq N$. Let $Y_{N,K}^{(s)}$ denote the number of nodes that set their timers to $s\Delta$ given that at least two nodes have done so and no node sets its timer before $s\Delta$.

Clearly, $\mathbf{Pr}\left[Y_{N,K}^{(s)} = l\right] = 0$, for $l = 0, 1$, because at least two nodes must be involved in a collision. From Baye's rule, we have, for $l \geq 2$,

$$\mathbf{Pr}\left[Y_{N,K}^{(s)} = l\right] = \frac{\mathbf{Pr}\left[X_{N,K}^{(s)} = l, X_{N,K}^{(s)} \geq 2, X_{N,K}^{(s-1)} = \cdots = X_{N,K}^{(0)} = 0\right]}{\mathbf{Pr}\left[X_{N,K}^{(s)} \geq 2, X_{N,K}^{(s-1)} = \cdots = X_{N,K}^{(0)} = 0\right]}. \quad (25)$$

Since $l \geq 2$, the numerator in (25) is simply equal to $\mathbf{Pr}[X_{N,K}^{(s)} = l, X_{N,K}^{(s-1)} = \cdots = X_{N,K}^{(0)} = 0]$. Note that this is nothing but the probability that $l$ out of $K$ metrics lie in the interval $\left[1-\sum_{j=0}^{s}\alpha_{N,K}[j], 1-\sum_{j=0}^{s-1}\alpha_{N,K}[j]\right)$ and the remaining $(K-l)$ metrics lie in the interval $\left[0, 1-\sum_{j=0}^{s}\alpha_{N,K}[j]\right)$. This probability is nothing but $\binom{K}{l}(\alpha_{N,K}[s])^l\left(1-\sum_{j=0}^{s}\alpha_{N,K}[j]\right)^{K-l}$.

On the other hand, using the law of total probability, the denominator in (25) is:

$$\begin{aligned}\mathbf{Pr}&\left[X_{N,K}^{(s)} \geq 2, X_{N,K}^{(s-1)} = \cdots = X_{N,K}^{(0)} = 0\right] \\ &= \mathbf{Pr}\left[X_{N,K}^{(s-1)} = \cdots = X_{N,K}^{(0)} = 0\right] \\ &\quad - \mathbf{Pr}\left[X_{N,K}^{(s)} = \cdots = X_{N,K}^{(0)} = 0\right] \\ &\quad - \mathbf{Pr}\left[X_{N,K}^{(s)} = 1, X_{N,K}^{(s-1)} = \cdots = X_{N,K}^{(0)} = 0\right].\end{aligned}$$

As before, each of the above probability terms can be written in terms of the stair lengths. The final expression for $\mathbf{Pr}[Y_{N,K}^{(s)} = l]$ can be shown to be

$$\begin{aligned}\mathbf{Pr}\left[Y_{N,K}^{(s)} = l\right] &= \binom{K}{l}(\alpha_{N,K}[s])^l\left(1-\sum_{j=0}^{s}\alpha_{N,K}[j]\right)^{K-l} \\ &\times \left[\left(1-\sum_{j=0}^{s-1}\alpha_{N,K}[j]\right)^K - K\alpha_{N,K}[s]\left(1-\sum_{j=0}^{s}\alpha_{N,K}[j]\right)^{K-1}\right. \\ &\qquad \left. - \left(1-\sum_{j=0}^{s}\alpha_{N,K}[j]\right)^K\right]^{-1}. \quad (26)\end{aligned}$$

We now show that the RVs $Y_{N,K}^{(s)}$ and $Y_{N-s,K}^{(0)}$ have the same distribution.

**Proposition** *5:* For $N \geq 0$ and $K \geq 2$,

1) $\mathbf{Pr}[Y_{N,K}^{(s)} = l] = \mathbf{Pr}[Y_{N-s,K}^{(0)} = l]$, for $0 \leq s \leq N$ and $0 \leq l \leq K$.
2) $\mathbf{E}[Y_{N,K}^{(s)}] = \mathbf{E}[Y_{N-s,K}^{(0)}]$, for $0 \leq s \leq N$.

*Proof:* We know that $\mathbf{Pr}\left[Y_{N,K}^{(s)} = l\right] = 0$, whenever $l = 0, 1$. Hence, for $l = 0, 1$, we have

$$\mathbf{Pr}\left[Y_{N,K}^{(s)} = l\right] = \mathbf{Pr}\left[Y_{N-s,K}^{(0)} = l\right], \quad (27)$$

for all $s \in \{0, 1, \ldots, N\}$. For $r \geq 2$, dividing the numerator and the denominator of (26) by $\left(1-\sum_{j=0}^{s-1}\alpha_{N,K}[j]\right)^K$, we get

$$\mathbf{Pr}\left[Y_{N,K}^{(s)} = l\right] = \frac{\binom{K}{l}\alpha'_{N,K}[s]^l\left(1-\alpha'_{N,K}[s]\right)^{K-l}}{1-K\alpha'_{N,K}[s]\left(1-\alpha'_{N,K}[s]\right)^{K-1} - \left(1-\alpha'_{N,K}[s]\right)^K}, \quad (28)$$

where $\alpha_{N,K}^{'}[s] = \alpha_{N,K}[s] / \left(1 - \sum_{j=0}^{s-1} \alpha_{N,K}[j]\right)$. Using Proposition 4, we know that $\alpha_{N,K}^{'}[s] = \alpha_{N-s,K}[0]$. Substituting this in (28), we get

$$\begin{aligned}&\mathbf{Pr}\left[Y_{N,K}^{(s)} = l\right]\\ &= \frac{\binom{K}{l}\alpha_{N-s,K}[0]^{l}\left(1-\alpha_{N-s,K}[0]\right)^{K-l}}{1-K\alpha_{N-s,K}[0](1-\alpha_{N-s,K}[0])^{K-1}-(1-\alpha_{N-s,K}[0])^{K}},\\ &= \mathbf{Pr}\left[Y_{N-s,K}^{(0)} = l\right].\end{aligned}$$

Since the probability mass functions of the two RVs $Y_{N,K}^{(s)}$ and $Y_{N-s,K}^{(0)}$ are the same, it follows that $\mathbf{E}[Y_{N,K}^{(s)}] = \mathbf{E}[Y_{N-s,K}^{(0)}]$. ■

**Theorem** *2:* In an $N$-level single-stage optimal timer, the average number of nodes $\mathbf{E}[Y_{N,K}^{(s)}]$ that set their timers to $s\Delta$, given that at least two timers have expired at $s\Delta$ and no timer has expired before $s\Delta$, lies in a narrow range given by:

$$2 \le \mathbf{E}[Y_{N,K}^{(s)}] \le \frac{e-1}{e-2} < 2.4, \quad \text{for } s = 0, 1, \dots, N. \tag{29}$$

*Proof:* The lower bound on $\mathbf{E}[Y_{N,K}^{(i)}]$ is easy to see since at least two nodes must collide in a collision. From Proposition 5, we know that $\mathbf{E}[Y_{N,K}^{(i)}] = \mathbf{E}[Y_{N-i,K}^{(0)}]$. Therefore, it is sufficient to show, for all $N \ge 0$ and $K \ge 2$, that $\mathbf{E}[Y_{N,K}^{(0)}]$ is upper bounded by $\frac{e-1}{e-2}$.

Substituting $j = 0$ in (1) and using (11), it can be shown that

$$P_{N,K} = \left(1 - \alpha_{N,K}[0]\right)^{K-1}. \tag{30}$$

From (26), we have $\mathbf{E}[Y_{N,K}^{(0)}] = \sum_{r=2}^{K} r\mathbf{Pr}\left[Y_{N,K}^{(0)} = r\right] = \frac{\left(1-(1-\alpha_{N,K}[0])^{K-1}\right)K\alpha_{N,K}[0]}{1-K\alpha_{N,K}[0](1-\alpha_{N,K}[0])^{K-1}-(1-\alpha_{N,K}[0])^{K}}$. Using (30), the above expression can be recast as

$$\mathbf{E}[Y_{N,K}^{(0)}] = \frac{\left(1-P_{N,K}\right)K\alpha_{N,K}[0]}{1-K\alpha_{N,K}[0]\,P_{N,K} - \left(1-\alpha_{N,K}[0]\right)P_{N,K}}. \tag{31}$$

Writing $\alpha_{N,K}[0] = \frac{1-P_{N-1,K}}{K-P_{N-1,K}}$ (from (1)), we get

$$\mathbf{E}[Y_{N,K}^{(0)}] = \left(1 - \xi^{(N)}(K)\right)^{-1}, \tag{32}$$

where

$$\xi^{(N)}(K) = \frac{K-1}{K}\left(\frac{1}{1-P_{N,K}} - \frac{1}{1-P_{N-1,K}}\right). \tag{33}$$

Thus, to upper bound $\mathbf{E}[Y_{N,K}^{(0)}]$, it suffices to upper bound $\xi^{(N)}(K)$, as done below. Clearly,

$$\xi^{(N)}(K) \le \sup_{N\ge 0}\ \sup_{K\ge 2} \frac{K-1}{K}\left(\frac{1}{1-P_{N,K}} - \frac{1}{1-P_{N-1,K}}\right). \tag{34}$$

Let $x = 1 - P_{N-1,K}$. From Proposition 2, we then have $P_{N,K} = \left(\frac{K-1}{K-1+x}\right)^{K-1}$. Further, since $x \in (0,1)$, we have

$$\xi^{(N)}(K) \le \sup_{K\ge 2}\ \sup_{x\in(0,1)} \left(\frac{1}{1-\left(\frac{K-1}{K-1+x}\right)^{K-1}} - \frac{1}{x}\right).$$

Since the function $\frac{1}{1-\left(\frac{K-1}{K-1+x}\right)^{K-1}} - \frac{1}{x}$ is a monotonically increasing function in $x$, for $x \in (0,1)$, we get

$$\xi^{(N)}(K) \le \sup_{K\ge 2} \frac{\left(\frac{K-1}{K}\right)^{K}}{1-\left(\frac{K-1}{K}\right)^{K-1}}. \tag{35}$$

Further, $\frac{\left(\frac{x-1}{x}\right)^{x}}{1-\left(\frac{x-1}{x}\right)^{x-1}}$ is an increasing function in $x$, for $x \in \mathbb{R}$ and $x \ge 2$. Therefore,

$$\xi^{(N)}(K) \le \lim_{K\to\infty} \frac{\left(\frac{K-1}{K}\right)^{K}}{1-\left(\frac{K-1}{K}\right)^{K-1}} = \frac{1}{e-1}.$$

Hence, $\mathbf{E}[Y_{N,K}^{(0)}] = \left(1 - \xi^{(N)}(K)\right)^{-1} \le \frac{e-1}{e-2} < 2.4$. ■

### B. *Proof of Theorem 1*

The first timer stage and the feedback that follows it require a total time of $(N+1)\Delta + F$. Three outcomes are possible at the end of the first stage: 1) *Success*: The scheme terminates and requires no more time. 2) *Idle*: It can be seen from the rescaling in (6) that the scheme requires an additional time of $T_2$, on average, to select. 3) *Collision*: The rescaling in (5) implies that, on average, $T_2$ more time is required to select.

Let $P_{N,2}^{(\text{idle})}$ and $P_{N,2}^{(\text{coll.})}$ denote the probabilities of idle and collision, respectively. Then, from the law of total expectation, we have

$$T_2 = (N+1)\Delta + F + \left(P_{N,2}^{(\text{coll.})} + P_{N,2}^{(\text{idle})}\right)T_2. \tag{36}$$

However, $P_{N,2}^{(\text{coll.})} + P_{N,2}^{(\text{idle})} = 1 - P_{N,2} = 1 - \frac{N+1}{N+2}$, from Proposition 2. Substituting this in (36) yields (7).

For $F > 0$, it is easy to see that $T_2$ is a strictly convex function in $N$, if $N \in \mathbb{R}$. It, thus, has a unique minimum, which can be shown to occur at $\sqrt{\frac{F}{\Delta}} - 1$ using the first order condition. If this is an integer, then it must be the global optimum. Else, one of the two integers nearest to it $\left\lceil\sqrt{\frac{F}{\Delta}} - 1\right\rceil$ or $\left\lfloor\sqrt{\frac{F}{\Delta}} - 1\right\rfloor$ must be optimal. For $F = 0$, (7) directly implies that $N^* = 0$.

### C. *Derivation of* (10)

Let $T_K$ denote the average selection time required to select the best node. The first stage and its feedback will require a time of $(N_I + 1)\Delta + F$. The scheme terminates after the first stage in case of a success, which occurs with probability $1 - P_{N_I,K}^{(\text{idle})} - P_{N_I,K}^{(\text{coll.})}$, where $P_{N_I,K}^{(\text{idle})}$ and $P_{N_I,K}^{(\text{coll.})}$ denote idle and collision probabilities, respectively.

From Theorem 2, we know that once a collision occurs, the number of nodes that have collided is close to two. Therefore, in case of a collision, an additional time of approximately $T_2$ is required to select the best node, on average. Similarly, in case of an idle outcome, an additional time of $T_K$ is required to select the best node, on average. Therefore, $T_K \approx (N_I+1)\Delta + F + P_{N_I,K}^{(\text{coll.})}T_2 + P_{N_I,K}^{(\text{idle})}T_K$. Substituting (7) in this expression gives the desired result.

### D. *Proof of Proposition 1*

In (10), we see that $T_K$ is an affine function of $T_2$ for any given $N_I$. Thus, the value of $N_C$ that minimizes $T_K$, for a given $N_I$, is the one that minimizes $T_2$. Using Theorem 1, the result follows.

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