E2 201 (Aug–Dec 2014)

Homework Assignment 1

Discussion: Friday, Aug. 15 Quiz: Friday, Aug. 22

1. Define a sequence (a_n) recursively as follows: $a_0 = 0$, and for $n \geq 1$, $a_n = \sqrt{2 + a_{n-1}}$. Prove that the sequence (a_n) converges to a limit. What is the limit?
[*Hint*: Guess an upper bound for the sequence, and show by induction that the sequence is monotonically increasing and bounded.]

2. Prove that if a sequence is convergent, then it is bounded (both above and below). Is the converse true?

3. Prove that $\limsup_{n\to\infty} a_n \leq L$ if and only if

   for each $\epsilon > 0$, the inequality $a_n < L + \epsilon$ holds for all sufficiently large n.

4. Let L be a real number. Show that $\lim_{n\to\infty} a_n = L$ if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$.

5. Determine the limsup and liminf of each of the following sequences:

   (a) a_n as defined in Problem 1.

   (b) $a_n = n \sin \frac{n\pi}{2}$ for all $n \geq 1$.

   (c) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$
   [The first term is $\frac{1}{2}$, the next two terms are $\frac{1}{3}, \frac{2}{3}$, the next three terms are $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$, and so on.]

6. For each of the statements below, prove or give a counterexample:

   (a) For A and B bounded subsets of $\mathbb{R}$, and $A + B \stackrel{\text{def}}{=} \{a + b : a \in A, b \in B\}$,
   $$\sup(A + B) = \sup A + \sup B.$$

   (b) For sequences (a_n) and (b_n),
   $$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

   (c) For convergent sequences (a_n) and (b_n),
   $$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

7. Let $\mathcal{X} = \{0, 1\}$. For a sequence $\mathbf{x}^n = (x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$, let $N_1(\mathbf{x}^n)$ denote the number of 1s in $\mathbf{x}^n$. Let $p(x)$ be a probability mass function (pmf) on $\mathcal{X}$, with $p(1) = p$. Let $X_1, X_2, X_3, \ldots$ be a sequence of iid random variables, each with pmf $p(x)$. For $\epsilon > 0$ and $n \geq 1$, define A_ϵ^n to be the typical set with respect to the pmf $p(x)$, and also define
   $$B_\epsilon^{(n)} = \left\{ \mathbf{x}^n \in \mathcal{X}^n : \left| \frac{1}{n} N_1(\mathbf{x}^n) - p \right| \leq \epsilon \right\}.$$

   (a) Is it true that $\Pr[A^{(n)} \cap B^{(n)}] \to 1$ as $n \to \infty$?

   (b) Show that $|A^{(n)} \cap B^{(n)}| \leq 2^{n(H(X)+\epsilon)}$ for all n.

   (c) Show that $|A^{(n)} \cap B^{(n)}| \geq (1/2)\, 2^{n(H(X)-\epsilon)}$ for all sufficiently large n.

8. Problem 3.13, Cover & Thomas, 2nd ed.