E2 201 (Aug–Dec 2014)

Homework Assignment 3

Discussion: Friday, Sept. 12 Quiz: Friday, Sept. 19

This assignment consists of two pages.

1. Let X and Y be random variables with joint pmf $p(x, y)$ tabulated as follows:

		Y	
X	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$
	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$

   (a) Give a good lower bound on the probability of error in estimating X from an observation of Y.

   (b) Find an estimator $\widehat{X} = g(Y)$ for which the probability of error Pr$[g(Y) \neq X]$ equals the lower bound obtained in (a).

2. Prove the Cèsaro mean theorem:

   If (a_n) is a sequence of real numbers that converges to a limit $L \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i = L$.

   Is the converse true?

3. Consider a time-invariant Markov chain on a finite-state space $\mathcal{X}$, with transition probability matrix P. Show that the Markov chain has Unif($\mathcal{X}$) as a stationary distribution if and only if the matrix P is doubly stochastic.

4. Problem 4.30, Cover & Thomas, 2nd ed.

5. A box contains two coins. Coin 1 is fair (Pr$[H] = 1/2$), and Coin 2 is biased so that Pr$[H] = p$ for some $p \neq 1/2$. One of these coins is chosen uniformly at random, i.e., Pr[Coin 1 is chosen] = Pr[Coin 2 is chosen] $= 1/2$. Let X denote the identity of the coin that is picked, so that $X = i$ if Coin i is picked. The chosen coin is tossed repeatedly, with $Y_1, Y_2, Y_3, \ldots$ denoting the outcomes of the sequence of coin tosses.

   (a) Is the random process $\mathbf{Y} = (Y_1, Y_2, Y_3, \ldots)$ stationary?

   (b) Is the random process $\mathbf{Y} = (Y_1, Y_2, Y_3, \ldots)$ i.i.d.?

   (c) What is the entropy rate, $\mathcal{H}(\mathbf{Y})$, of $\mathbf{Y}$?
   [*Hint*: First argue that $\lim \frac{1}{n} H(Y^n)$ is equal to $\lim \frac{1}{n} H(Y^n|X)$.]

   (d) Does $-\frac{1}{n} \log p(Y^n)$ converge to $\mathcal{H}(\mathbf{Y})$ in probability?

6. Consider the same set-up as in the previous problem. We are interested in binary $n : k$ block codes for the source $\mathbf{Y} = (Y_1, Y_2, Y_3, \ldots)$. Recall our definition of $k(n, \delta)$ to be the least integer k for which there exists a binary $n : k$ block source code (f, g) for $\mathbf{Y}$, with $P_{\text{err}} \leq \delta$. Show that for $0 \leq \delta < 1/2$, we have

$$\lim_{n\to\infty} \frac{k(n, \delta)}{n} = 1.$$

   Why does $\frac{k(n,\delta)}{n}$ not converge to $\mathcal{H}(\mathbf{Y})$?

7. Let $\mathcal{C}$ be a D-ary prefix code with codeword lengths $\ell_1, \ell_2, \ldots, \ell_m$, and let T be any code tree for $\mathcal{C}$.

   (a) Show that $\sum_{i=1}^{m} D^{-\ell_i} = 1$ iff every leaf of T is a codeword of $\mathcal{C}$.

   (b) Show that if $\sum_{i=1}^{m} D^{-\ell_i} < 1$, then there exist arbitrarily long sequences of code symbols in $\mathcal{D}^*$ which cannot be resolved (parsed) into sequences of codewords.

8. Problem 5.30, Cover & Thomas, 2nd ed.