E2 201 (Aug–Dec 2014)

Homework Assignment 5

Discussion: Friday, Oct. 17 Quiz: Friday, Oct. 24

This assignment consists of two pages.

1. Recall that a discrete memoryless channel $(\mathcal{X}, \{p(y \mid x)\}, \mathcal{Y})$ is *weakly symmetric* if its transition probability matrix P has the property that the rows of P are all permutations of each other, and the columns of P all have the same sum. Prove that the capacity of a weakly symmetric channel is

$$C = \log_2 |\mathcal{Y}| - H(\mathbf{r})$$

where $\mathbf{r}$ is the first row of P. What is the input distribution that achieves capacity?

2. Problem 7.5, Cover & Thomas, 2nd ed.

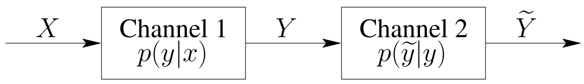
3. Problem 7.8, Cover & Thomas, 2nd ed.

4. Problem 7.9, Cover & Thomas, 2nd ed.

5. Problem 7.28(a), Cover & Thomas, 2nd ed.

6. Let $(\mathcal{X}, \{p(y|x)\}, \mathcal{Y})$ and $(\mathcal{Y}, \{p(\widetilde{y}|y)\}, \widetilde{\mathcal{Y}})$ be two discrete memoryless channels, which we call Channel 1 and Channel 2, respectively. Let C_i denote the capacity of Channel i, for $i = 1, 2$.

   (a) The output of Channel 1 is fed directly into the input of Channel 2, as depicted in the figure below.

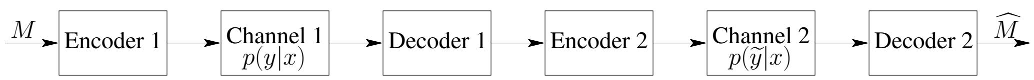


   Let $\widetilde{C}$ denote the capacity of the overall channel between X and $\widetilde{Y}$. In general, what can you say about the relationship between $\widetilde{C}$, C_1 and C_2?

   (b) Determine $\widetilde{C}$ when Channels 1 and 2 are binary symmetric channels with crossover probability p.

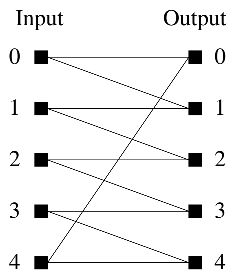
   (c) Determine $\widetilde{C}$ when Channel 1 is a DMC with binary output alphabet $\mathcal{Y} = \{0, 1\}$ and capacity C_1, and Channel 2 is a binary erasure channel with erasure probability ϵ.
   [*Hint*: Express $I(X; \widetilde{Y})$ in terms of $I(X; Y)$ and ϵ.]

   (d) In part (a), suppose that instead of directly connecting the output of the first channel to the input of the second channel, we are allowed encoders at the inputs of both channels and decoders at their outputs — see figure below.



   What now is the supremum of achievable rates of reliable information transmission across the overall end-to-end system from M to $\widehat{M}$?

7. Consider the discrete memoryless channel with input and output alphabets $\mathcal{X} = \mathcal{Y} = \{0, 1, 2, 3, 4\}$, depicted in the figure below. All transitions shown in the figure have probability $1/2$.



An (M, n) code $\mathcal{C}$ for this channel is called a *zero-error code* if $P_e(\mathcal{C}) = 0$. For example, the $(2, 1)$ code $\mathcal{C}^{(1)} = \{0, 2\}$ is a zero-error code for the channel: the two codewords have no channel outputs in common.

As usual, the rate of an (M, n) code is $\frac{1}{n} \log_2 M$. The supremum of the rates of zero-error codes is called the *zero-error capacity* of the channel, denoted by C_0.

(a) Justify the bounds $1 \leq \mathsf{C}_0 \leq \log_2 5 - 1$.

(b) Give a lower bound on C_0 that is strictly better than 1. [*Hint*: Consider codes of blocklength 2.]

[*Remark*: Computing the zero-error capacity of a channel is difficult in general. For the specific channel in this problem, Lovász (1979) proved that the zero-error capacity is in fact $\frac{1}{2} \log_2 5$.]

8. Let $(\mathcal{X}, \{p(y|x)\}, \mathcal{Y})$ be a discrete memoryless channel. In class, we defined for $\rho \in [0, 1]$,

$$\beta(\rho) = -\log_2 \sum_{y \in \mathcal{Y}} \left[\sum_{x \in \mathcal{X}} p(x) p(y|x)^{\frac{1}{1+\rho}} \right]^{1+\rho}.$$

The random coding error exponent is given by

$$\mathcal{E}(R) = \max_{0 \leq \rho \leq 1} \max_{\{p(x)\}} [\beta(\rho) - \rho R],$$

the inner maximum being over all probability distributions on $\mathcal{X}$.

(a) Verify that $\beta(\rho) \geq 0$ for all $\rho \in [0, 1]$, and $\beta'(0) = I(X; Y)$.

(b) For the channel depicted below, determine the capacity C and the error exponent $\mathcal{E}(R)$ in closed form.