E2 201 (Aug–Dec 2014)

Homework Assignment 6

Discussion: Friday, Oct. 31   Quiz: Friday, Nov. 7

This assignment consists of two pages.

1. The usual memoryless binary symmetric channel may be viewed as a channel of the form

$$Y_n = X_n \oplus Z_n, \;\; n = 1, 2, 3, \ldots,$$

where $X_n \in \{0, 1\}$ for all n, (Z_n) is an i.i.d. Ber(p) sequence independent of (X_n), and $\oplus$ denotes modulo-2 addition. We know that the capacity of this channel is $1 - H(p)$.

Now, consider a binary symmetric channel with memory, described as follows. At the beginning of time, a noise bit Z is randomly chosen according to a Ber(p) distribution, and thereafter *fixed* for all time:

$$Y_n = X_n \oplus Z, \;\; n = 1, 2, 3, \ldots.$$

Determine the capacity of this channel, where we now define capacity to be

$$C = \lim_{n\to\infty} \frac{1}{n} C_n,$$

with $C_n = \max_{p(x^n)} I(X^n; Y^n)$, the maximum being taken over all pmfs $p(x^n)$ on the input alphabet $\{0, 1\}^n$.

2. Consider the function

$$f(x) = \begin{cases} \frac{1}{x(\ln x)^2} & \text{if } x > e \\ 0 & \text{otherwise} \end{cases}$$

   (a) Verify that f is a density function.

   (b) Determine $h(f)$.

3. (a) Let $S \subseteq \mathbb{R}$, functions $r_i(x)$, $i = 1, 2, \ldots, m$, and constants α_i, $i = 1, 2, \ldots, m$, all be given. Let $\mathfrak{F}$ represent the family of all densities f with the following properties:

   - $f(x) = 0$ for all $x \notin S$, i.e., Supp$(f) \subseteq S$
   - $\int_S f(x)\, dx = 1$
   - $\int_S r_i(x) f(x)\, dx = \alpha_i$, for $i = 1, 2, \ldots, m$.

   Now, let $f^*(x) = \exp(\lambda_0 + \sum_{i=1}^m \lambda_i r_i(x))$, where $\lambda_0, \lambda_1, \ldots, \lambda_m$ are chosen so that f^* belongs to the family $\mathfrak{F}$. (Assume that such λ_i's can indeed be chosen.) Prove that f^* uniquely maximizes the differential entropy $h(f)$ over all densities $f \in \mathfrak{F}$.

   (b) Let X be a non-negative random variable with $\mathbb{E}[X] = \mu$, where $\mu > 0$ is a fixed constant. Show that $h(X) \leq h(Z)$, where $Z \sim$ EXP($\frac{1}{\mu}$), i.e., Z has density $f(z) = \frac{1}{\mu} e^{-z/\mu}$, $z \geq 0$.

   (c) What happens if we remove the non-negative requirement on X in part (b)? In other words, what is $\sup h(X)$, where the supremum is taken over all continuous random variables X with fixed mean $\mathbb{E}[X] = \mu$.

4. Problem 8.8, Cover & Thomas, 2nd ed.

5. Problem 9.3, Cover & Thomas, 2nd ed.

6. Problem 9.7, Cover & Thomas, 2nd ed.

7. Let $Y = X + Z$, where X is $\mathcal{N}(0, P)$, and Z is a random variable independent of X, with mean 0 and variance σ^2. Given an observation of Y, we want to estimate X.

   (a) Among *linear* estimators $\hat{X}(Y)$ of the form $aY + b$, determine the estimator that minimizes the mean squared error $\mathbb{E}\big[\big(X - \hat{X}(Y)\big)^2\big]$. What is the resulting minimum mean squared error?

   (b) Give an upper bound on $h(X|Y)$ using the result of part (a). When is this bound tight?

8. Consider any continuous channel with additive noise as follows:

$$Y_n = X_n + Z_n,$$

   where the noise sequence (Z_n) is iid with mean 0 and variance σ^2, and (Z_n) is independent of (X_n). Prove that the capacity, $C(P)$, of this channel, under an average input power constraint P, is greater than or equal to $\frac{1}{2}\log(1 + P/\sigma^2)$, with equality iff the additive noise is iid $\mathcal{N}(0, \sigma^2)$.

   Thus, among additive noise channels of fixed noise variance, the Gaussian channel has the least capacity. [*Hint*: Use the result of Problem 7(b).]