E2 201 (Aug–Dec 2014)

Homework Assignment 8

Discussion: Friday, Nov. 28 No quiz

This assignment consists of two pages.

1. The rate-distortion theorem was proved in class for discrete memoryless sources and bounded, normal distortion measures. Modify the proof of the theorem so that it applies even to distortion measures $d : \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}_+ \cup \{+\infty\}$ that are not bounded but instead satisfy the following weaker condition:

   there exists an $\hat{x}_0 \in \hat{\mathcal{X}}$ and a constant $D_0 < \infty$ such that $d(x, \hat{x}_0) \leq D_0$ for all $x \in \mathcal{X}$.

   (See the next problem for an example of such a distortion measure.)
   [*Hint*: Only the achievability proof needs to be modified. To the randomly generated reconstruction codebook, add an extra codeword $(\hat{x}_0, \hat{x}_0, \ldots, \hat{x}_0)$.]

2. [Problem 10.7, Cover & Thomas, 2nd ed.] Let $X \sim$ Bernoulli$(\frac{1}{2})$, and let the reconstruction alphabet be $\hat{\mathcal{X}} = \{0, 1, \varepsilon\}$, where ε represents an erasure symbol. Consider the distortion measure given by

$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x} \\ 1 & \text{if } x \in \{0,1\} \text{ and } \hat{x} = \varepsilon \\ +\infty & \text{if } (x, \hat{x}) = (0,1) \text{ or } (1,0) \end{cases}$$

   Determine the rate-distortion function for this source. Suggest a simple scheme to achieve any point on the rate-distortion curve.

3. *Product source.* Let X and Y be two independent sources with alphabets $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$ be the respective reproduction alphabets, and let distortion measures $d_{\mathcal{X}} : \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}_+$ and $d_{\mathcal{Y}} : \mathcal{Y} \times \hat{\mathcal{Y}} \to \mathbb{R}^+$ be given. Let the rate-distortion functions for X and Y be denoted by $R_x(D)$ and $R_y(D)$, respectively.

   Consider the product source (X, Y) with distortion measure $d : (\mathcal{X} \times \mathcal{Y}) \times (\hat{\mathcal{X}} \times \hat{\mathcal{Y}}) \to \mathbb{R}_+$ defined as follows:

$$d((x, y), (\hat{x}, \hat{y})) = d_{\mathcal{X}}(x, \hat{x}) + d_{\mathcal{Y}}(y, \hat{y})$$

   Prove that the rate-distortion function for the product source with distortion measure d is given by

$$R(D) = \min_{D_1 + D_2 = D} (R_x(D_1) + R_y(D_2))$$

   [*Hint*: Use the fact that $I(X, Y ; \hat{X}, \hat{Y}) \geq I(X; \hat{X}) + I(Y; \hat{Y})$ when X and Y are independent (why is this true?)]

4. Problem 10.17, Cover & Thomas, 2nd ed.

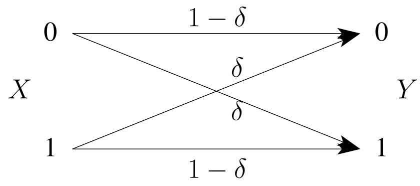
5. Problem 10.20, Cover & Thomas, 2nd ed.

6. [Problem 15.34, Cover & Thomas, 2nd ed.] Let X and Y be independent Bernoulli(p) random variables, and let $U = XY$, $V = X + Y$, the operations being perfomed modulo 2 (thus, $1 + 1 = 0$). This induces a joint distribution on (U, V). Let (U_i, V_i), $i = 1, 2, 3, \ldots$ be drawn iid according to this joint distribution. Encoder 1 describes U^n at rate R_1, and Encoder 2 describes V^n at rate R_2.

   (a) Determine the Slepian-Wolf rate region for recovering (U^n, V^n) with probability of error going to 0 as $n \to \infty$.

(b) What is the residual uncertainty that the receiver has about (X^n, Y^n), i.e., what is $H(X^n, Y^n \mid U^n, V^n)$?

7. Let X be the input and Y be the output of a binary symmetric channel with crossover probability δ.



Assume that $X \sim \text{Bernoulli}(\frac{1}{2})$, which induces a joint distribution, $p(x, y)$, on (X, Y). Determine the achievable rate region for the distributed source coding (Slepian-Wolf) problem for a discrete memoryless source that produces an i.i.d. source sequence $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), \ldots$, with $(X_i, Y_i) \sim p(x, y)$.