AN APPLICATION OF RAMSEY THEORY TO CODING FOR THE OPTICAL CHANNEL*

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Abstract. In this paper, we analyze bi-infinite sequences over the alphabet $\{0, 1, \ldots, q-1\}$, for an arbitrary $q \geq 2$, that satisfy the q-ary ghost pulse (qGP) constraint. A sequence, $\mathbf{x} = (x_k)_{k\in\mathbb{Z}} \in \{0, 1, \ldots, q-1\}^{\mathbb{Z}}$ satisfies the qGP constraint if for all $k, l, m \in \mathbb{Z}$ such that x_k, x_l and x_m are non-zero and equal, x_{k+l-m} is also non-zero. This constraint arises in the context of coding for communication over a fiber optic medium. We show using techniques from Ramsey theory that if $\mathbf{x}$ satisfies the qGP constraint, then the set $\mathrm{supp}(\mathbf{x}) = \{l \in \mathbb{Z} : \ x_l \neq 0\}$ is the disjoint union of cosets of some subgroup, $k\mathbb{Z}$, of $\mathbb{Z}$, and a set of zero density. We provide much sharper results in the special cases of $q = 2$ and $q = 3$. In the former case, we show that the corresponding binary ghost pulse constraint has zero capacity, and based on our results for the latter case, we conjecture that the capacity of the ternary ghost pulse constraint is also zero.

Key words. Optical communication, constrained coding, ghost pulse constraints, Ramsey theory

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1. Introduction. In this paper, we study the effect of a class of constraints which we call "ghost pulse" constraints imposed on sequences over a finite alphabet. Throughout the paper, we shall follow the standard convention of using $\mathbb{Z}$ to denote the set of all integers and $\mathbb{N}$ to denote the set of positive integers. Also, given $m, n \in \mathbb{Z}$, we shall take $[m, n]$ to be the set $\{k \in \mathbb{Z} : m \leq k \leq n\}$. Given an integer $q \geq 2$, let $\mathcal{A}_q = \{0, 1 \ldots, q-1\}$. For $\mathbf{x} = (x_k)_{k\in\mathbb{Z}} \in \mathcal{A}_q{}^{\mathbb{Z}}$, we define the support of $\mathbf{x}$ to be $\mathrm{supp}(\mathbf{x}) = \{k \in \mathbb{Z} : x_k \neq 0\}$.

DEFINITION 1.1 (q-ary Ghost Pulse (qGP) constraint). *A sequence $\mathbf{x} \in \mathcal{A}_q{}^{\mathbb{Z}}$ satisfies the qGP constraint if for all $k, l, m \in \mathrm{supp}(\mathbf{x})$ (k, l, m not necessarily distinct) such that $x_k = x_l = x_m$, we also have $k + l - m \in \mathrm{supp}(\mathbf{x})$.*

We shall denote by $\mathcal{T}_q$ the set of all $\mathbf{x} \in \mathcal{A}_q{}^{\mathbb{Z}}$ that satisfy the qGP constraint. Furthermore, we shall use $\mathcal{S}_q$ to denote the set of all $\mathbf{y} \in \{0, 1\}^{\mathbb{Z}}$ such that there exists an $\mathbf{x} \in \mathcal{T}_q$ with $\mathrm{supp}(\mathbf{x}) = \mathrm{supp}(\mathbf{y})$. The object of this paper is to study the sequences in $\mathcal{S}_q$, particularly in the cases when q is 2 or 3. When $q = 2$, we refer to the corresponding constraint as the *binary ghost pulse (BGP)* constraint, and when $q = 3$, the corresponding constraint is called the *ternary ghost pulse (TGP)* constraint.

These ghost pulse constraints arise in the context of coding for communication over a fiber optic medium. In a typical optical communication scenario, a train of light pulses corresponding to a sequence of M bits is sent across the fiber optic medium that constitutes the optical channel. Each bit in the sequence is allocated a time slot of duration T, and a 1 or 0 is marked by the presence or absence of a pulse in that

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time slot. A nonlinear phenomenon known as four-wave mixing causes a transfer of energy from triples of pulses in '1' slots into certain '0' slots, creating spurious pulses called *ghost pulses*. It has been observed [1],[9] that the interaction of pulses in the kth, lth and mth time slots (k, l, m need not all be distinct) in the pulse train pumps energy into the $(k+l-m)$th time slot. If this slot did not originally contain a pulse, *i.e.*, if the $(k+l-m)$th bit was a 0 in the original M-bit sequence, then the transfer of energy creates a ghost pulse in the slot, thus changing the original 0 to a 1.

The formation of ghost pulses may be modeled as follows: let $b_0 b_1 \ldots b_{M-1}$, $b_i \in \{0,1\}$, be the binary sequence corresponding to the transmitted train of pulses. If we have 1's in positions k, l, m (not necessarily all distinct) in this sequence, *i.e.*, $b_k = b_l = b_m = 1$, and if $b_{k+l-m} = 0$, then the formation of a ghost pulse converts b_{k+l-m} to a 1. Note that if $b_0 b_1 \ldots b_{M-1}$ were a subblock of a sequence $\mathbf{x} \in \mathcal{S}_2$, then no ghost pulses would be formed, since if i is a position where a ghost pulse could potentially be created, then b_i is already a 1 by the definition of the BGP constraint. So, one way of eliminating the formation of ghost pulses when transmitting an arbitrary data sequence $b_0 b_1 \ldots b_{M-1}$, $b_i \in \{0,1\}$, is to first encode the data sequence into a sequence $c_0 c_1 \ldots c_{N-1}$ that is a subblock of some $\mathbf{x} \in \mathcal{S}_2$.

The efficiency of any coding scheme using subblocks of BGP-constrained sequences as codewords is limited by the *capacity*, $h(\mathcal{S}_2)$, of $\mathcal{S}_2$, which is defined as

$$h(\mathcal{S}_2) = \lim_{n \to \infty} \frac{\log_2 |\mathcal{B}_{2,n}|}{n} \tag{1}$$

where $\mathcal{B}_{2,n}$ denotes the set of all length-n subblocks of sequences in $\mathcal{S}_2$. The closer $h(\mathcal{S}_2)$ is to 1, the more efficient are the coding schemes based on BGP-constrained sequences. However, it is easily shown that $h(\mathcal{S}_2) = 0$, as a consequence of the following simple characterization of sequences in $\mathcal{S}_2$.

THEOREM 1.2. *A binary sequence* $\mathbf{x}$ *is in* $\mathcal{S}_2$ *if and only if* $\text{supp}(\mathbf{x}) = \emptyset$ *or* $\text{supp}(\mathbf{x}) = a + k\mathbb{Z}$ *for some* $a, k \in \mathbb{Z}$.

Proof: It is clear from the definition of the BGP constraint that if $\mathbf{x} \in \{0,1\}^{\mathbb{Z}}$ is such that $\text{supp}(\mathbf{x}) = \emptyset$ or $\text{supp}(\mathbf{x}) = a + k\mathbb{Z}$, then $\mathbf{x} \in \mathcal{S}_2$. For the converse, suppose that $\mathbf{x} \in \mathcal{S}_2$ is such that $\text{supp}(\mathbf{x}) \neq \emptyset$. Take any $a \in \text{supp}(\mathbf{x})$ and let $H = \text{supp}(\mathbf{x}) - a = \{k - a : x_k \neq 0\}$. It is easily verified that H is a subgroup of $\mathbb{Z}$, and hence, $H = k\mathbb{Z}$ for some integer k. Thus, $\text{supp}(\mathbf{x}) = a + H = a + k\mathbb{Z}$. □

COROLLARY 1.3. $h(\mathcal{S}_2) = 0$.

Proof: It follows from the above theorem that $|\mathcal{B}_{2,n}| = O(n^2)$, which implies that $h(\mathcal{S}_2) = 0$. □

Thus, any coding scheme based on BGP-constrained sequences is bound to be inefficient in terms of rate. So, we need to consider alternative approaches to dealing with the ghost pulse problem.

One approach that has been suggested to mitigate the formation of ghost pulses is to apply, at the transmitter end, a phase shift of π to some of the pulses in the '1' time slots [7], [2]. The interaction of pulses with different phases suppresses the formation of ghost pulses at certain locations due to destructive interference. However, three pulses with the same phase can still interact to create ghost pulses. We can effectively think of this phase modulation technique as converting a binary sequence $b_0b_1\ldots b_{N-1}$ into a ternary sequence, $c_0c_1\ldots c_{N-1}$, with $c_i \in \{-1,0,1\}$, such that $b_i = |c_i|$ for all $i \in \{0,1,\ldots,N-1\}$. As a first-order approximation of the true situation, we shall assume that the only case in which a ghost pulse is formed is when we have $c_k = c_l = c_m = 1$ or $c_k = c_l = c_m = -1$, and $c_{k+l-m} = 0$.

Now, if $\mathbf{x} \in \{-1,0,1\}^{\mathbb{Z}}$ is a sequence satisfying the TGP constraint[1], and if $c_0c_1\ldots c_{N-1}$ is a subblock of $\mathbf{x}$, then by the first-order approximation stated above, $c_0c_1\ldots c_{N-1}$ can be transmitted without error across the optical channel. Thus, finite-length subblocks of sequences in $\mathcal{T}_3$ can be used as codewords for encoding binary data sequences.

However, there is a catch. In reality, an optical receiver can only detect the amplitude of the optical signal at the channel output, not its phase. What this means is that if the transmitted ternary sequence was $c_0c_1\ldots c_{N-1}$, then the receiver only sees the sequence $|c_0|, |c_1|, \ldots, |c_{N-1}|$, *i.e.*, the receiver cannot distinguish a 1 from a -1. As a result, we cannot use two ternary sequences that differ only in phase (*i.e.*, only in sign) to encode two different binary data sequences.

So, the proper procedure to encode and transmit a finite-length binary data sequence $a_0a_1\ldots a_{M-1}$ is to first encode it with a subblock $b_0b_1\ldots b_{N-1}$ of some sequence in $\mathcal{S}_3$ which, before transmission, is converted to a subblock, $c_0c_1\ldots c_{N-1}$, of some sequence in $\mathcal{T}_3$. At the channel output, the receiver detects the sequence $b_0b_1\ldots b_{N-1}$ which can be decoded correctly to recover $a_0a_1\ldots a_{M-1}$. We thus have a rather unusual coding problem because even though the sequence being transmitted is a ternary sequence, the alphabet used for the encoding of information is effectively binary.

Consequently, the efficiency of any coding scheme that uses TGP-constrained sequences is limited by the capacity, $h(\mathcal{S}_3)$, of the set $\mathcal{S}_3$, which is defined analogously to (1) as follows:

$$h(\mathcal{S}_3) = \lim_{n\to\infty} \frac{\log_2 |\mathcal{B}_{3,n}|}{n} \tag{2}$$

where $\mathcal{B}_{3,n}$ denotes the set of all length-n subblocks of sequences in $\mathcal{S}_3$. It should be pointed out that the existence of the limits in (1) and (2) follows by standard arguments from the following fact (*cf.* [6, Chapter 4]): if $a_1, a_2, \ldots$ is a sequence of non-negative numbers such that $a_{m+n} \le a_m + a_n$ for all $m, n \ge 1$, then $\lim_{n\to\infty} a_n/n$ exists and equals $\inf_{n\ge 1} a_n/n$.

In this paper, we analyze the structure of the sequences in $\mathcal{S}_3$ in an attempt to provide a simple characterization for them along the lines of Theorem 1.2, which could then be used to determine $h(\mathcal{S}_3)$. Unfortunately, the TGP constraint is much harder to analyze than its binary counterpart. It is actually instructive to study the qGP constraint for arbitrary $q \ge 2$ as it provides useful insight into the ternary case. In fact, extension to the q-ary alphabet allows for an unexpectedly simple and elegant analysis based on results drawn from the branch of mathematics known as Ramsey theory.

Using results from Ramsey theory, we show in Theorem 3.1 that any sequence $\mathbf{y} \in \mathcal{S}_q$ is "almost periodic", in the sense that it can be transformed into a periodic sequence by changing a relatively sparse subset of the 1's to 0's. More precisely, we show that if $\mathbf{y} \in \mathcal{S}_q$, then there exists a subset $N(\mathbf{y}) \subset \mathrm{supp}(\mathbf{y})$, such that $N(\mathbf{y})$ has

[1] In Definition 1.1, ternary sequences are defined over the alphabet $\{0,1,2\}$. We simply identify the symbol 2 with -1.

density[2] 0, and $\mathrm{supp}(\mathbf{y}) \setminus N(\mathbf{y})$ is a union of cosets of some subgroup, $k\mathbb{Z}$, of $\mathbb{Z}$. For sequences in $\mathcal{S}_3$, we make this result much stronger by showing in Theorem 4.4 that any $\mathbf{y} \in \mathcal{S}_3$ can be made periodic by changing at most two 1's to 0's. In fact, this theorem provides a simple and complete description of the aperiodic sequences in $\mathcal{S}_3$. We also provide a useful characterization (Theorem 4.1) of periodic sequences in $\mathcal{S}_3$, which we use to completely describe all such sequences of prime period (Theorem 4.3). Based on these results and some numerical evidence, we conjecture that $h(\mathcal{S}_3) = 0$.

The remainder of the paper is organized as follows. In Section 2, we provide the background from Ramsey theory needed for our proofs. Section 3 contains our analysis of qGP-constrained sequences, and Section 4 presents the analysis for TGP-constrained sequences. In Section 5, we present some numerical evidence in support of our conjecture that $h(\mathcal{S}_3) = 0$.

2. Some Ramsey Theory. Given a set I and a positive integer k, we refer to any function $\chi : I \to [1, k]$ as a *k-coloring* of I. Observe that if $V_j = \{i \in I : \chi(i) = j\}$, then the sets V_j, $j = 1, 2, \ldots, k$, form a partition of I, which we shall call the *chromatic partition* (with respect to the coloring χ) of I. The sets V_j are often called the *color classes* of χ. A subset $J \subset I$ is said to be *monochromatic* (wrt χ) if $J \subset V_j$ for some $j \in [1, k]$.

Ramsey theory is a branch of combinatorics which deals with structure which is preserved under partitions [3]. A typical result from Ramsey theory guarantees that when some set I is finitely colored, then some structure of the set I appears in monochromatic form. One of the classic results of Ramsey theory is the following theorem due to Schur [4, Chapter 3, Theorem 1].

THEOREM 2.1 (Schur's theorem). *Given a $k \in \mathbb{N}$, there exists an $N(k) \in \mathbb{N}$ such that for all $n \geq N(k)$, every k-coloring of $[1, n]$ contains a monochromatic solution to $x + y = z$.*

To put it another way, Schur's theorem states that given a $k \in \mathbb{N}$, for all sufficiently large n, if we partition $[1, n]$ into k subsets, $V_1, V_2 \ldots, V_k$, then there exist $x, y, z \in V_i$ for some i, that satisfy $x + y = z$. The smallest integer $N(k)$ for which the statement of Schur's theorem holds is referred to as the kth Schur number, and is denoted by $S(k)$. The exact value of $S(k)$ is only known for $k = 1, 2, 3, 4$: $S(1) = 2$, $S(2) = 5$, $S(3) = 14$, $S(4) = 45$ [8, Sequence A030126].

Another well-known result from Ramsey theory, known as van der Waerden's theorem [4, Chapter 2, Theorem 1], guarantees the existence of arbitrarily long monochromatic arithmetic progressions in any k-coloring of the integers. Recall that an arithmetic progression (A.P.) of length l is a subset of the integers of the form $\{a + id : i = 0, 1, \ldots, l - 1\}$, for some $a \in \mathbb{Z}$ and $d \in \mathbb{N}$. We shall require a stronger form of van der Waerden's theorem, to state which we need the following definition.

DEFINITION 2.2 (Upper density). *Given $I \subset \mathbb{Z}$, the upper density of I is defined to be*

$$\overline{d}(I) = \limsup_{n\to\infty} \frac{|I \cap [-n, n]|}{2n + 1}$$

[2] Density is defined in Section 2.

Note that if $I_1, I_2, \dots I_k$ form a (finite) partition of $I \subset \mathbb{Z}$, then $\overline{d}(I) = \sum_{i=1}^{k} \overline{d}(I_i)$. In particular, if χ is a k-coloring of $\mathbb{Z}$, and $\{V_1, V_2, \dots, V_k\}$ is the corresponding chromatic partition of $\mathbb{Z}$, then $\sum_{j=1}^{k} \overline{d}(V_j) = 1$, since $\overline{d}(\mathbb{Z}) = 1$. Therefore, in any k-coloring, χ, of $\mathbb{Z}$, at least one of the color classes, V_j, $j = 1, 2, \dots, k$, of χ must have positive upper density. Thus, van der Waerden's theorem is a consequence of the following stronger result, known as Szemerédi's theorem [4, Chapter 2, p. 43].

Theorem 2.3 (Szemerédi's theorem). *If $I \subset \mathbb{Z}$ has positive upper density (*i.e., *$\overline{d}(I) > 0$), then for any $l \in \mathbb{N}$, I contains an A.P. of length l.*

The relevance of colorings to the study of qGP constrained sequences can be seen from the following simple lemma.

Lemma 2.4. *A binary sequence $\mathbf{y}$ is in $\mathcal{S}_q$ if and only if there exists a $(q-1)$-coloring, χ, of $\operatorname{supp}(\mathbf{y})$ such that whenever $k, l, m \in \operatorname{supp}(\mathbf{y})$ satisfy $\chi(k) = \chi(l) = \chi(m)$, then $k + l - m \in \operatorname{supp}(\mathbf{y})$.*

Proof: If $\mathbf{y}$ is in $\mathcal{S}_q$, then there exists an $\mathbf{x} \in \mathcal{T}_q$ with $\operatorname{supp}(\mathbf{x}) = \operatorname{supp}(\mathbf{y})$. For $k \in \operatorname{supp}(\mathbf{y})$, let $\chi(k) = x_k$. Then, by definition of the qGP constraint, χ is a $(q-1)$-coloring of $\operatorname{supp}(\mathbf{y})$ with the required property.

Conversely, if $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$ is such that χ is a $(q-1)$-coloring of $\operatorname{supp}(\mathbf{y})$ as in the statement of the lemma, then let $\mathbf{x} = (x_k)_{k\in\mathbb{Z}}$ be the sequence defined by $x_k = 0$ if $k \notin \operatorname{supp}(\mathbf{y})$, and $x_k = \chi(k)$ if $k \in \operatorname{supp}(\mathbf{y})$. Thus, $\operatorname{supp}(\mathbf{x}) = \operatorname{supp}(\mathbf{y})$, and by definition of the qGP constraint, $\mathbf{x} \in \mathcal{T}_q$. □

3. The qGP Constraint. We shall use the results from Ramsey theory provided in the previous section to prove our main result on qGP-constrained sequences, which we state next.

Theorem 3.1. *For $q \geq 2$, if $\mathbf{x} \in \mathcal{S}_q$ or $\mathbf{x} \in \mathcal{T}_q$, then there exists an integer $k \geq 0$ and a set $I \subset [0, k-1]$, both depending on $\mathbf{x}$, such that*

$$\bigcup_{i\in I} (k\mathbb{Z} + i) \subset \operatorname{supp}(\mathbf{x})$$

and

$$\overline{d}\left(\operatorname{supp}(\mathbf{x}) \setminus \bigcup_{i\in I} (k\mathbb{Z} + i)\right) = 0$$

In other words, outside a set of density 0, $\operatorname{supp}(\mathbf{x})$ is a union of cosets of some subgroup, $k\mathbb{Z}$, of $\mathbb{Z}$. It is enough to prove this theorem for $\mathbf{x} \in \mathcal{T}_q$, since for any $\mathbf{y} \in \mathcal{S}_q$, there exists an $\mathbf{x} \in \mathcal{T}_q$ with $\operatorname{supp}(\mathbf{x}) = \operatorname{supp}(\mathbf{y})$. Our proof of the theorem relies on the following proposition, which shows that if $\mathbf{x} \in \mathcal{T}_q$ is such that $\operatorname{supp}(\mathbf{x})$ contains a sufficiently large number of consecutive terms of $a + k\mathbb{Z}$ for some $a, k \in \mathbb{Z}$, then it must in fact contain all of $a + k\mathbb{Z}$. Recall that for $q \in \mathbb{N}$, $S(q)$ is the qth Schur number.

Proposition 3.2. *For $q \geq 2$, if $\mathbf{x} \in \mathcal{T}_q$ is such that $\operatorname{supp}(\mathbf{x})$ contains a $S(q-1)$-term A.P., $\{a + jk : 1 \leq j \leq S(q-1)\}$ for some $a, k \in \mathbb{Z}$, then $a + k\mathbb{Z} \subset \operatorname{supp}(\mathbf{x})$.*

Proof: Suppose that $\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \in \mathcal{T}_q$ is such that $\mathrm{supp}(\mathbf{x})$ contains $a + jk$, $1 \le j \le S(q-1)$. We shall show that a and $a + (S(q-1)+1)k$ are also in $\mathrm{supp}(\mathbf{x})$, so that the result then follows by induction.

Define a $(q-1)$-coloring, χ, of $[1, S(q-1)]$ via $\chi(j) = x_{a+jk}$ for $j \in [1, S(q-1)]$. This is indeed a $(q-1)$-coloring since $a + jk \in \mathrm{supp}(\mathbf{x})$, and hence, $x_{a+jk} \neq 0$ for $j \in [1, S(q-1)]$. By Schur's theorem, there exist $r, s, t \in [1, S(q-1)]$ such that $r + s = t$ and $\chi(r) = \chi(s) = \chi(t)$, or equivalently, $x_{a+rk} = x_{a+sk} = x_{a+tk}$. But now, $(a + rk) + (a + sk) - (a + tk) = a$, so that by definition of the qGP constraint, $a \in \mathrm{supp}(\mathbf{x})$ as well.

A similar argument using the coloring of $[1, S(q-1)]$ defined by

$$\hat{\chi}(j) = \chi(S(q-1) + 1 - j) = x_{a+(S(q-1)+1-j)k}$$

proves that $a + (S(q-1)+1)k \in \mathrm{supp}(\mathbf{x})$, which completes the proof of the proposition. ☐

For $\mathbf{x} \in \mathcal{A}_q{}^{\mathbb{Z}}$ and $j \in [0, q-1]$, we define

$$V_j(\mathbf{x}) = \{k \in \mathbb{Z} : x_k = j\} \tag{3}$$

Thus, the sets $V_j(\mathbf{x})$, $j \in [0, q-1]$, constitute a partition of $\mathbb{Z}$, while $V_j(\mathbf{x})$, $j \in [1, q-1]$, is a partition of $\mathrm{supp}(\mathbf{x})$. Note that $\mathbf{x} \in \mathcal{T}_q$ if and only if the sets $V_j(\mathbf{x})$ satisfy the following condition: for any $j \in [1, q-1]$, if $k, l, m \in V_j(\mathbf{x})$, then $k + l - m \in \mathrm{supp}(\mathbf{x})$. The following is a useful corollary to the above proposition.

COROLLARY 3.3. *Let $q \ge 2$ and $N = \lceil \frac{S(q-1)+2}{3} \rceil$. If $\mathbf{x} \in \mathcal{T}_q$ is such that for some $j \in [1, q-1]$, $V_j(\mathbf{x})$ contains an N-term A.P. $\{a + \ell k : 0 \le \ell \le N-1\}$ for some $a, k \in \mathbb{Z}$, then $a + k\mathbb{Z} \subset \mathrm{supp}(\mathbf{x})$.*

Proof: Suppose that $a, k \in \mathbb{Z}$ are such that $\{a + \ell k : 0 \le \ell \le N-1\} \subset V_j(\mathbf{x})$ for some $j \in [1, q-1]$. Note that for $0 \le \ell \le N-2$, $a + (N+\ell)k = (a + (N-1)k) + (a + (N-1)k) - (a + (N-2-\ell)k)$. Since $a + (N-1)k, a + (N-2-\ell)k \in V_j(\mathbf{x})$, it follows from the definition of the qGP constraint that $a + (N+\ell)k \in \mathrm{supp}(\mathbf{x})$.

Similarly, for $1 \le \ell \le N-1$, $a - \ell k = a + a - (a + \ell k)$, and hence, $a - \ell k \in \mathrm{supp}(\mathbf{x})$ as well. Thus, $\mathrm{supp}(\mathbf{x})$ contains the $(3N+2)$-term A.P. $\{a + \ell k : -(N-1) \le \ell \le 2(N-1)\}$. Since $3N + 2 \ge S(q-1)$, the result follows from Proposition 3.2. ☐

The next lemma forms the crux of the proof of Theorem 3.1.

LEMMA 3.4. *For $\mathbf{x} \in \mathcal{T}_q$, if $\overline{d}(V_j(\mathbf{x})) > 0$ for some $j \in [1, q-1]$, then there exists a $k_j \in \mathbb{N}$ such that if we let*

$$I_j = \{i \in [0, k_j - 1] : \ |V_j(\mathbf{x}) \cap (k_j\mathbb{Z} + i)| > 0\}$$

then

$$V_j(\mathbf{x}) \subset \bigcup_{i \in I_j} (k_j\mathbb{Z} + i) \subset \mathrm{supp}(\mathbf{x}).$$

Proof: By the definition of I_j, it is obvious that for any j, $V_j \subset \bigcup_{i\in I_j}(k_j\mathbb{Z} + i)$. So, we shall show that if $\overline{d}(V_j(\mathbf{x})) > 0$ for some $j \in [1, q-1]$, then there exists $k_j \neq 0$

such that, with I_j as defined above, $\bigcup_{i\in I_j}(k_j\mathbb{Z}+i)\subset \mathrm{supp}(\mathbf{x})$. Indeed, it suffices to show that for each $i\in I_j$, $k_j\mathbb{Z}+i\subset \mathrm{supp}(\mathbf{x})$.

Without loss of generality, we may assume that $\overline{d}(V_1(\mathbf{x}))>0$. By Szemerédi's theorem, $V_1(\mathbf{x})$ contains an $S(q-1)$-term A.P. $\{a+jk_1 : 1\le j\le S(q-1)\}$ for some $a\in\mathbb{Z}$ and $k_1\in\mathbb{N}$. Now, take any $i\in I_1$, where I_1 is as in the statement of the lemma. We need to show that $k_1\mathbb{Z}+i\subset \mathrm{supp}(\mathbf{x})$. Since i is in I_1, there exists an $m\in\mathbb{Z}$ such that $i+mk_1\in V_1(\mathbf{x})$. But now, for any $j\in[1,S(q-1)]$, since $a+jk_1\in V_1(\mathbf{x})$, the qGP constraint implies that $(i+mk_1)+(a+jk_1)-(a+k_1)=i+(m+j-1)k_1$ is in $\mathrm{supp}(\mathbf{x})$. We thus have an $S(q-1)$-term A.P. $\{i+(m+j-1)k_1 : 1\le j\le S(q-1)\}$ in $\mathrm{supp}(\mathbf{x})$, and hence by Proposition 3.2, $i+k_1\mathbb{Z}\subset \mathrm{supp}(\mathbf{x})$, as desired. ◻

We are now in a position to prove Theorem 3.1. Given $\mathbf{x}\in\mathcal{T}_q$, we shall let $J_1=\{j\in[1,q-1]:\overline{d}(V_j(\mathbf{x}))>0\}$ and $J_2=\{j\in[1,q-1]:\overline{d}(V_j(\mathbf{x}))=0\}$. Also, let $P(\mathbf{x})=\bigcup_{j\in J_1}V_j(\mathbf{x})$ and $N(\mathbf{x})=\bigcup_{j\in J_2}V_j(\mathbf{x})$. Clearly, $P(\mathbf{x}),N(\mathbf{x})$ form a partition of $\mathrm{supp}(\mathbf{x})$ with $\overline{d}(P(\mathbf{x}))>0$, if $P(\mathbf{x})\ne\emptyset$, and $\overline{d}(N(\mathbf{x}))=0$.

Proof of Theorem 3.1: As mentioned earlier, it suffices to prove the theorem for $\mathbf{x}\in\mathcal{T}_q$. If $\overline{d}(\mathrm{supp}(\mathbf{x}))=0$, then we may take $k=0$. So, we may assume that $\overline{d}(\mathrm{supp}(\mathbf{x}))>0$, so that $J_1\ne\emptyset$. For each $j\in J_1$, let k_j and I_j be as in the statement of Lemma 3.4, and let $k=\mathrm{lcm}\{k_j : j\in J_1\}$ be the least common multiple of the k_j's. Since

$$k_j\mathbb{Z}+i=\bigcup_{\ell=0}^{k/k_j-1}(k\mathbb{Z}+\ell k_j+i),$$

if we define $\widehat{I}_j$ to be $\{\ell k_j+i : i\in I_j, \ell\in[0,k/k_j-1]\}$, then

$$\bigcup_{i\in I_j}(k_j\mathbb{Z}+i)=\bigcup_{i\in\widehat{I}_j}(k\mathbb{Z}+i).$$

Therefore, for each $j\in J_1$,

$$V_j(\mathbf{x})\subset\bigcup_{i\in\widehat{I}_j}(k_j\mathbb{Z}+i)\subset \mathrm{supp}(\mathbf{x}).$$

Now, taking $I=\bigcup_{j\in J_1}\widehat{I}_j$, we see that

$$P(\mathbf{x})\subset\bigcup_{i\in I}(k\mathbb{Z}+i)\subset \mathrm{supp}(\mathbf{x})$$

Finally,

$$\mathrm{supp}(\mathbf{x})\setminus\bigcup_{i\in I}(k\mathbb{Z}+i)\subset \mathrm{supp}(\mathbf{x})\setminus P(\mathbf{x})=N(\mathbf{x})$$

from which it follows that

$$\overline{d}\left(\mathrm{supp}(\mathbf{x})\setminus\bigcup_{i\in I}(k\mathbb{Z}+i)\right)=0,$$

which completes the proof of the theorem. □

It is straightforward to see that the set I in the above proof is in fact the set $\{i \in [0, k-1] : |P(\mathbf{x}) \cap (k\mathbb{Z} + i)| > 0\}$.

Corollary 3.5. *If $\mathbf{x} \in \mathcal{T}_q$ is such that $N(\mathbf{x}) = \emptyset$, then the sequence $\mathbf{y} \in \mathcal{S}_q$ with* $\mathrm{supp}(\mathbf{y}) = \mathrm{supp}(\mathbf{x})$ *is a periodic sequence.*

Proof: If $\mathrm{supp}(\mathbf{x}) = \emptyset$, then $\mathbf{x}$, as well as the corresponding $\mathbf{y} \in \mathcal{S}_q$, is simply the all-zeros sequence, $0^{\mathbb{Z}}$, which is periodic. If $\mathrm{supp}(\mathbf{x}) \neq \emptyset$, but $N(\mathbf{x}) = \emptyset$, then $\mathrm{supp}(\mathbf{x}) = P(\mathbf{x})$. So, as in the proof of the above theorem, there exists a $k \in \mathbb{N}$ and an $I \subset [0, k-1]$ such that

$$\mathrm{supp}(\mathbf{x}) \subset \bigcup_{i \in I} (k\mathbb{Z} + i) \subset \mathrm{supp}(\mathbf{x}).$$

Hence, $\mathrm{supp}(\mathbf{x}) = \bigcup_{i \in I} (k\mathbb{Z} + i)$. It now follows that the corresponding $\mathbf{y} \in \mathcal{S}_q$ is periodic with period k. □

It appears to be difficult to strengthen Theorem 3.1 any further, for example, to give a complete description of the sequences that are allowed to be in $\mathcal{T}_q$ or $\mathcal{S}_q$ for arbitrary q. However, for $q = 3$, which is our main case of interest, we can do much better than Theorem 3.1, as we show in the next section.

4. The TGP Constraint. In this section, we provide a means of characterizing the binary sequences that are in $\mathcal{S}_3$. Separate characterizations are provided for binary sequences that are periodic, and for those that are not. Recall that $\mathbf{y} = (y_m)_{m \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ is *periodic* if there exists a $k \in \mathbb{N}$ such that $y_m = y_{m+k}$ for all $m \in \mathbb{Z}$. The integer k is referred to as a *period* of $\mathbf{y}$. The *fundamental period* of a periodic sequence, $\mathbf{y}$, is the smallest $k \in \mathbb{N}$ that is a period of $\mathbf{y}$. Note that if $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$ is not the all-zeros sequence $0^{\mathbb{Z}}$, then $\mathbf{y}$ is periodic if and only if there exists a $k \in \mathbb{N}$ and a non-empty $I \subset [0, k-1]$ such that $\mathrm{supp}(\mathbf{y}) = \bigcup_{i \in I} (k\mathbb{Z} + i)$. The following theorem shows that a non-zero periodic sequence having k as a period is in $\mathcal{S}_3$ if and only if it satisfies a certain "modulo-k" ternary ghost pulse constraint, in a manner made precise in the statement of the theorem.

Theorem 4.1. *Let $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$, $\mathbf{y} \neq 0^{\mathbb{Z}}$, be periodic, so that there exists a $k \in \mathbb{N}$ and a non-empty $I \subset [0, k-1]$ such that* $\mathrm{supp}(\mathbf{y}) = \bigcup_{i \in I} (k\mathbb{Z} + i)$. *Then, $\mathbf{y} \in \mathcal{S}_3$ if and only if there exists a 2-coloring, χ, of I such that whenever $i_1, i_2, i_3 \in I$ satisfy $\chi(i_1) = \chi(i_2) = \chi(i_3)$, then $i_1 + i_2 - i_3 \mod k \in I$.*

Since $I \equiv \mathrm{supp}(\mathbf{y}) \pmod{k}$, a comparison of the statement of the above theorem with that of Lemma 2.4 highlights the "modulo-k" nature of the ternary ghost pulse constraint imposed on periodic sequences in $\mathcal{S}_3$. The modulo-k connection can be made explicit in terms of the definition given below. For $k \in \mathbb{N}$, let $\mathbb{Z}/k$ denote the group of integers modulo k, *i.e.*, $\mathbb{Z}/k$ is the set $[0, k-1]$ equipped with the operation of modulo-k addition.

Definition 4.2 (TGP-coloring). *For $k \in \mathbb{N}$ and $I \subset \mathbb{Z}/k$, a TGP-coloring of I is a 2-coloring, χ, of I such that whenever $i_1, i_2, i_3 \in I$ satisfy $\chi(i_1) = \chi(i_2) = \chi(i_3)$,* *then* $i_1 + i_2 - i_3 \in I$.

We would like to clarify that whenever we write $I \subset \mathbb{Z}/k$, we tacitly assume that I gets equipped with the same operation as $\mathbb{Z}/k$. So, in the above definition, $i_1+i_2-i_3$ is in fact taken modulo k.

A subset $I \subset \mathbb{Z}/k$ is said to be *TGP-colorable* if there exists a TGP-coloring of I. Thus, we may re-state Theorem 4.1 as follows: let $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$, $\mathbf{y} \neq 0^{\mathbb{Z}}$, be periodic, so that there exists a $k \in \mathbb{N}$ and a non-empty $I \subset [0, k-1]$ such that $\operatorname{supp}(\mathbf{y}) = \bigcup_{i \in I}(k\mathbb{Z} + i)$. Then, $\mathbf{y} \in \mathcal{S}_3$ if and only if $I \subset \mathbb{Z}/k$ is TGP-colorable.

While the $\mathbb{Z}/k$ TGP-colorability condition is easier to check than the full-blown TGP condition, it is still very hard in practice to verify that this condition holds for an arbitrary $I \subset \mathbb{Z}/k$. However, if p is a prime, then we can determine precisely which subsets of $\mathbb{Z}/p$ are TGP-colorable.

THEOREM 4.3. *Let p be prime. Then, $I \subset \mathbb{Z}/p$ is TGP-colorable if and only if one of the following holds:*
(i) $|I| \leq 2$*;*
(ii) $I = [0, p-1]$*;*
(iii) $p = 5$ *and* $|I| = 4$.

For non-prime k, the problem of determining all the subsets of $\mathbb{Z}/k$ that are TGP-colorable remains open. In other words, we do not yet have an easily verifiable characterization for periodic sequences in $\mathcal{S}_3$ whose fundamental period is non-prime. In Table 5.1 in the next section, we list the number of TGP-colorable subsets of $\mathbb{Z}/k$ for non-prime $k \leq 20$, obtained by means of an exhaustive computer search.

Luckily, the problem of determining which *aperiodic* sequences are in $\mathcal{S}_3$ turns out to be a lot easier. There is a simple characterization of such sequences, which is presented in the following theorem.

THEOREM 4.4. *Let $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$ be an aperiodic sequence. Then, $\mathbf{y} \in \mathcal{S}_3$ if and only if one of the following conditions holds:*
(i) $1 \leq |\operatorname{supp}(\mathbf{y})| \leq 2$*;*
(ii) there exists a $k \in \mathbb{N}$ and an $i \in [0, k-1]$ such that $\operatorname{supp}(\mathbf{y}) = (k\mathbb{Z} + i) \cup V$*, with $V = \{j\}$ for some $j \in \mathbb{Z}$, $j \not\equiv i \pmod{k}$;*
(iii) there exists a $t \in \mathbb{N}$ and an $i \in [0, 3t-1]$ such that $\operatorname{supp}(\mathbf{y}) = (3t\mathbb{Z} + i) \cup V$*, with $|V| = 2$ and $V \equiv \{t+i, 2t+i\} \pmod{3t}$.*

The remainder of this section is devoted to the proofs of Theorems 4.1, 4.3 and 4.4. For the purpose of the proofs, we shall find it convenient to define the function $\pi : \{0,1,2\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ as follows: for $\mathbf{x} \in \{0,1,2\}^{\mathbb{Z}}$, $\pi(\mathbf{x})$ is the unique $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$ such that $\operatorname{supp}(\mathbf{y}) = \operatorname{supp}(\mathbf{x})$. Observe that $\pi(\mathcal{T}_3) = \mathcal{S}_3$.

4.1. Periodic sequences in $\mathcal{S}_3$. The proof of Theorem 4.1 is based on the following lemma.

LEMMA 4.5. *Let $\mathbf{y} \in \mathcal{S}_3$ be such that* $\operatorname{supp}(\mathbf{y}) = \bigcup_{i \in I}(k\mathbb{Z} + i)$*, for some $k \in \mathbb{N}$ and a non-empty $I \subset [0, k-1]$. Then, there exists an $\mathbf{x} \in \mathcal{T}_3$ such that $\pi(\mathbf{x}) = \mathbf{y}$, $V_1(\mathbf{x}) = \bigcup_{i \in I_1}(k\mathbb{Z} + i)$ and $V_2(\mathbf{x}) = \bigcup_{i \in I_2}(k\mathbb{Z} + i)$ for some partition $\{I_1, I_2\}$ of I.*

Proof: Let $\mathbf{y}$, k and I be as in the statement of the lemma, and let $\overline{I} = [0, k-1] \setminus I$. Since $\mathbf{y} \in \mathcal{S}_3$, there exists a $\mathbf{z} \in \mathcal{T}_3$ such that $\pi(\mathbf{z}) = \mathbf{y}$. For each $i \in I$, $k\mathbb{Z} + i \subset \text{supp}(\mathbf{z}) = V_1(\mathbf{z}) \cup V_2(\mathbf{z})$, where $V_1(\mathbf{z}), V_2(\mathbf{z})$ are as defined in (3). Let $I_1 = \{i \in I : \ |(kZ + i) \cap V_1(\mathbf{z})| > 0\}$, and let $I_2 = I \setminus I_1$. Thus, $\{I_1, I_2\}$ is a partition of I. Now, define $\mathbf{x} = (x_j)_{j \in \mathbb{Z}} \in \{0,1,2\}^{\mathbb{Z}}$ as follows:

$$x_j = \begin{cases} 0 & \forall\, j \in k\mathbb{Z} + i,\ i \notin I \\ 1 & \forall\, j \in k\mathbb{Z} + i,\ i \in I_1 \\ 2 & \forall\, j \in k\mathbb{Z} + i,\ i \in I_2 \end{cases}$$

Clearly, $\pi(\mathbf{x}) = \mathbf{y}$, $V_1(\mathbf{x}) = \bigcup_{i \in I_1} (k\mathbb{Z} + i)$ and $V_2(\mathbf{x}) = \bigcup_{i \in I_2} (k\mathbb{Z} + i)$. We shall show that $\mathbf{x} \in \mathcal{T}_3$.

Suppose, to the contrary, that $\mathbf{x} \notin \mathcal{T}_3$, so that there exist $p, q, r \in V_1(\mathbf{x})$ or $p, q, r \in V_2(\mathbf{x})$ such that $p + q - r \notin \text{supp}(\mathbf{x})$. Without loss of generality, we may assume that $p, q, r \in V_1(\mathbf{x})$ and $p + q - r \notin \text{supp}(\mathbf{x}) = \bigcup_{i \in I} (k\mathbb{Z} + i)$. Thus, $p + q - r \in \bigcup_{i \in \overline{I}} (k\mathbb{Z} + i)$, so that $p + q - r \mod k \notin I$.

Since $p, q, r \in V_1(x)$, $p \equiv i_1 \pmod{k}$, $q \equiv i_2 \pmod{k}$ and $r \equiv i_3 \pmod{k}$ for some $i_1, i_2, i_3 \in I_1$. Now, by definition of I_1, for each $i \in I_1$, there exists $t \in V_1(\mathbf{z})$ such that $t \equiv i \pmod{k}$. In particular, there exist $p', q', r' \in V_1(\mathbf{z})$ such that $p' \equiv i_1 \pmod{k}$, $q' \equiv i_2 \pmod{k}$ and $r' \equiv i_3 \pmod{k}$. In other words, $p' \equiv p \pmod{k}$, $q' \equiv q \pmod{k}$ and $r' \equiv r \pmod{k}$. Now, since $\mathbf{z} \in \mathcal{T}_3$, $p' + q' - r' \in \text{supp}(\mathbf{z}) = \bigcup_{i \in I} (k\mathbb{Z} + i)$, and hence, $p' + q' - r' \mod k \in I$. But since $p' + q' - r' \equiv p + q - r \pmod{k}$, this contradicts $p + q - r \mod k \notin I$. Thus, $\mathbf{x}$ must be in $\mathcal{T}_3$, thus proving the lemma. $\square$

Proof of Theorem 4.1: Let $\mathbf{y}$, k and I be as in the statement of the theorem. Suppose that there exists a 2-coloring, $\chi : I \to \{1, 2\}$, such that whenever $i_1, i_2, i_3 \in I$ satisfy $\chi(i_1) = \chi(i_2) = \chi(i_3)$, then $i_1 + i_2 - i_3 \mod k \in I$. Define $\mathbf{x} = (x_j)_{j \in \mathbb{Z}} \in \{0,1,2\}^{\mathbb{Z}}$ as follows: $x_j = \chi(j \mod k)$ if $j \mod k \in I$, and $x_j = 0$ otherwise. It is easy to verify that $\mathbf{x} \in \mathcal{T}_3$, and $\text{supp}(\mathbf{x}) = \bigcup_{i \in I} (k\mathbb{Z} + i) = \text{supp}(\mathbf{y})$, so that $\mathbf{y} \in \mathcal{S}_3$.

If $\mathbf{y} \in \mathcal{S}_3$, then let $\mathbf{x}$, I_1 and I_2 be as in the statement of Lemma 4.5. Define the 2-coloring, χ, of I as follows: $\chi(j) = 1$ if $j \in I_1$, and $\chi(j) = 2$ if $j \in I_2$. From the fact that $\mathbf{x} \in \mathcal{T}_3$, it follows that χ has the property stated in the theorem. $\square$

Our next goal is to provide a proof for Theorem 4.3. As is often the case, one direction of the theorem, namely the sufficiency of conditions (i), (ii) or (iii), is easy to prove. Indeed, if $|I| \leq 2$, then any injective 2-coloring, χ, of I is a TGP-coloring. If $I = [0, p-1]$, then we may take $\chi(i) = 1$ for all $i \in I$ to be the required TGP-coloring. Finally, if $p = 5$, then one may actually verify by hand that each of the five 4-subsets of $\mathbb{Z}/5$ is in fact TGP-colorable.

We must now prove that one of conditions (i)–(iii) of Theorem 4.3 is necessary for the existence of a TGP-coloring of $I \subset \mathbb{Z}/p$. In fact, we need only prove that if $I \subset \mathbb{Z}/p$ is TGP-colorable, and $|I| \geq 3$, then one of conditions (ii) and (iii) must hold. Note that $|I| \geq 3$ requires that $p \geq 3$, so we need not deal with $p = 2$.

Given a 2-coloring, χ, of $I \subset \mathbb{Z}/p$, we shall let $\{I_1, I_2\}$ denote the corresponding chromatic partition of I, *i.e.*, for $j = 1, 2$, $I_j = \{i \in I : \chi(i) = j\}$. Observe that if $I \subset \mathbb{Z}/p$ is TGP-colorable, and $|I| \geq 3$, then at least one of the following must be true:

(a) there exists a TGP-coloring, χ, of I such that $|I_1| \geq 2$ and $|I_2| \leq |I_1| - 2$;

(b) there exists a TGP-coloring, χ, of I such that $|I_1| \geq 2$ and $|I_2| = |I_1| - 1$;
(c) there exists a TGP-coloring, χ, of I such that $|I_1| \geq 2$ and $|I_2| = |I_1|$.

We shall analyze each of these cases separately, and show that if (a) or (b) is true, then $I = \mathbb{Z}/p$, and if (c) is true, then condition (iii) of Theorem 4.3 holds.

The following lemma is the $\mathbb{Z}/p$-equivalent of Corollary 3.3 (for the case $q = 3$), and is the core ingredient in our proofs.

LEMMA 4.6. *Let χ be a TGP-coloring of $I \subset \mathbb{Z}/p$, for prime $p \geq 3$, and let $\{I_1, I_2\}$ be the corresponding chromatic partition of I. If either I_1 or I_2 contains a 3-term A.P. $\{a, a+d, a+2d\}$, for some $a, d \in \mathbb{Z}/p$, $d \neq 0$, then $I = \mathbb{Z}/p$.*

Proof: Let $\mathbf{x} = (x_k)_{k\in\mathbb{Z}} \in \{0,1,2\}^{\mathbb{Z}}$ be the periodic sequence defined as follows: $x_j = \chi(j \mod k)$ if $j \mod k \in I$, and $x_j = 0$ otherwise. From the conditions of the lemma, and recalling that the Schur number $S(2)$ equals 5, we see that $\mathbf{x}$ satisfies the hypotheses of Corollary 3.3 for $q = 3$. Hence, $a + d\mathbb{Z} \subset \operatorname{supp}(\mathbf{x})$, from which it follows that I contains the set $K = \{a + jd \mod p : j \in \mathbb{Z}\}$. But, note that $H = \{jd \mod p : j \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}/p$, and K is a coset of H. Since the only non-empty subgroup of $\mathbb{Z}/p$ is $\mathbb{Z}/p$ itself, it follows that $K = \mathbb{Z}/p$, which proves the lemma. □

The following proposition takes care of case (a) above.

PROPOSITION 4.7. *Let χ be a TGP-coloring of $I \subset \mathbb{Z}/p$, for prime $p \geq 3$, such that $|I_1| \geq 2$ and $|I_2| \leq |I_1| - 2$. Then, $I = \mathbb{Z}/p$.*

Proof: Let $I_1 = \{i_1, i_2, \ldots, i_m\}$, $m \geq 2$, so that $|I_2| \leq m - 2$. Let $j_k = 2i_1 - i_k$, $k = 2, 3, \ldots, m$. Since χ is a TGP-coloring of I, $j_k \in I = I_1 \cup I_2$, for $k = 2, 3, \ldots, m$. Now, note that the j_k's are all distinct, since $j_k = j_l$ implies that $i_k = i_l$. Since there are $m - 1$ distinct j_k's and $|I_2| \leq m - 2$, there exists a $k \in [2, m]$ such that $j_k \in I_1$. But now, $\{i_k, i_1, j_k\}$ is a 3-term A.P. in I_1, and hence by Lemma 4.6, $I = \mathbb{Z}/p$. □

To deal with the remaining cases (b) and (c), we need the following lemma.

LEMMA 4.8. *Let I_1, I_2 be disjoint subsets of $\mathbb{Z}/p$, for prime $p \geq 3$, with $|I_1| \geq 2$ and $|I_2| = |I_1| - 1$. If for all pairs of distinct $x_i, x_j \in I_1$, we have $2x_i - x_j \in I_2$, then $p = 3$ and $I_1 \cup I_2 = \mathbb{Z}/3$.*

Proof: Let $I_1 = \{x_1, x_2, \ldots, x_m\}$ for some $m \geq 2$, so that $|I_2| = m - 1$. Clearly, $m < p$, as otherwise I_1, I_2 cannot be disjoint. For each $i \in [1, m]$, define $Y_i = \{2x_i - x_j : j \in [1, m], j \neq i\}$. Similarly, for each $j \in [1, m]$, define $Z_j = \{2x_i - x_j : i \in [1, m], i \neq j\}$. By assumption, $Y_i, Z_j \subset I_2$ for all $i, j \in [1, m]$. Furthermore, note that for any fixed i, the elements of Y_i are all distinct in $\mathbb{Z}/p$. Therefore, for any $i \in [1, m]$, $|Y_i| = m - 1$, and hence, $Y_i = I_2$. Similarly, $Z_j = I_2$ for any $j \in [1, m]$.

Now, fix an arbitrary $a \in I_2$. For any $i \in [1, m]$, since $Y_i = I_2$, there exists a *unique* $j \in [1, m]$, $j \neq i$, such that $2x_i - x_j = a$. Therefore, we can define a function $\sigma : [1, m] \to [1, m]$ as follows: $\sigma(i)$ is the unique j such that $2x_i - x_j = a$. We shall show that σ is a bijection which would imply that it is a permutation of $[1, m]$.

To show injectivity, we observe that if $\sigma(i) = \sigma(k) = j$ for some $i \neq k$, then we would have $2x_i - x_j = 2x_k - x_j$ for $i \neq k$. This would imply that $|Z_j| < m - 1$, which is impossible. To see that σ is surjective, take any $j \in [1, m]$, and note that $a \in Z_j$, as $Z_j = I_2$. Hence, there exists an $i \in [1, m]$ such that $2x_i - x_j = a$.

Thus, σ is a bijection, and hence, a permutation of $[1, m]$. Now, consider the m equations $2x_i - x_{\sigma(i)} = a$, $i = 1, 2, \ldots, m$. Adding all these equations, we find

$$\begin{aligned} ma &= 2\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} x_{\sigma(i)} \\ &= 2\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} x_i \qquad (4) \\ &= \sum_{i=1}^{m} x_i \end{aligned}$$

with the equality in (4) following from the fact that σ is a permutation of $[1, m]$.

Since $2 \leq m < p$, there exists an $m^{-1} \in \mathbb{Z}/p$ such that $mm^{-1} = 1$. Therefore, $a = m^{-1} \sum_{i=1}^{m} x_i$. Since our choice of $a \in I_2$ was arbitrary, it follows that I_2 consists of the single element $a = m^{-1} \sum_{i=1}^{m} x_i$. Therefore, $m - 1 = |I_2| = 1$, which shows that $m = 2$.

We thus have $I_1 = \{x_1, x_2\}$ and $I_2 = \{a\}$, so that by assumption, $2x_1 - x_2 = 2x_2 - x_1 = a$. But this means that $3(x_1 - x_2) = 0$ which, since $x_1 \neq x_2$, implies that $p = 3$. Therefore, since I_1 and I_2 are disjoint, we must have $I_1 \cup I_2 = \mathbb{Z}/p$. □

We can now readily dispose of case (b).

PROPOSITION 4.9. *Let χ be a TGP-coloring of $I \subset \mathbb{Z}/p$, for prime $p \geq 3$, such that $|I_1| \geq 2$ and $|I_2| = |I_1| - 1$. Then, $I = \mathbb{Z}/p$.*

Proof: Since χ is a TGP-coloring of I, for any $x_i, x_j \in I_1$, $2x_i - x_j$ is either in I_1 or I_2. If for some pair of distinct $x_i, x_j \in I_1$, $2x_i - x_j$ is also in I_1, then $\{x_j, x_i, 2x_j - x_i\}$ is a 3-term A.P. in I_1, and hence $I = \mathbb{Z}/p$ by Lemma 4.6. If not, Lemma 4.8 applies, which also shows that $I = \mathbb{Z}/p$. □

We are now left with case (c) which, unfortunately, requires some work. We start with the following simple lemma.

LEMMA 4.10. *Let χ be a TGP-coloring of $I \subset \mathbb{Z}/p$, for prime $p \geq 3$, such that $|I_1| = |I_2|$. Then, neither I_1 nor I_2 can contain a 3-term A.P.*

Proof: If either I_1 or I_2 contains a 3-term A.P., then by Lemma 4.6, $I = \mathbb{Z}/p$, implying that $|I| = p$, which is an odd number. However, $|I| = |I_1| + |I_2| = 2|I_1|$ is even. □

Thus, if x_1, x_2 is any pair of distinct elements in I_1, then $2x_2 - x_1 \in I_2$, for otherwise $\{x_1, x_2, 2x_2 - x_1\}$ would be a 3-term A.P. in I_1. By the same reasoning, $2y_2 - y_1 \in I_1$ for all $y_1, y_2 \in I_2$, $y_1 \neq y_2$.

The special case when $|I_1| = |I_2| = 2$ is straightforward, so we dispose of that first.

LEMMA 4.11. *If χ is a TGP-coloring of $I \subset \mathbb{Z}/p$, for prime $p \geq 3$, such that $|I_1| = |I_2| = 2$, then, $p = 5$.*

Proof: Let $I_1 = \{a, b\}$ for some $a, b \in [0, p-1]$, $a \neq b$. Since a, b are distinct elements in I_1, we must have $2a - b, 2b - a \in I_2$. Note that $2a - b \neq 2b - a$, for otherwise, we would have $3(a-b) = 0$, which would imply that $p = 3$, which contradicts $p \geq |I_1| + |I_2| = 4$. Therefore, $I_2 = \{2a - b, 2b - a\}$.

Now, since $2a - b \neq 2b - a$, we must have $2(2a - b) - (2b - a) = 5a - 4b \in I_1$. So, either $5a - 4b = a$ or $5a - 4b = b$. In the former case, we would get $4(a - b) = 0$, which would imply that $p|4$, which is impossible as $p \neq 2$. Therefore, we must have $5a - 4b = b$, from which we obtain $5(a - b) = 0$, and hence $p = 5$ as desired. □

The analysis of case (c) and, as a result, the proof of Theorem 4.3 would be complete if we can show that there cannot exist any TGP-coloring of $I \subset \mathbb{Z}/p$ such that $|I_1| = |I_2| \geq 3$. We show this in Proposition 4.14 below, but we need some development before we can prove the proposition.

Given a TGP-coloring, χ, of $I \subset \mathbb{Z}/p$, such that $|I_1| = |I_2| \geq 2$, we define certain functions $f, g : I_1 \to I_2$ as follows. Let $I_1 = \{x_1, x_2, \ldots, x_m\}$ and $I_2 = \{y_1, y_2, \ldots, y_m\}$, $m \geq 2$. For each $i \in [1, m]$, define the sets $Y_i = \{2x_j - x_i : j \in [1, m], j \neq i\}$, and $Z_i = \{2x_i - x_j : j \in [1, m], j \neq i\}$, so that $Y_i, Z_i \subset I_2$. For any fixed i, all the elements of Y_i are distinct, and hence $|Y_i| = m - 1$. Since $|I_2| = m$, there is precisely one element in $I_2 \setminus Y_i$, and we shall denote this element by $f(x_i)$. Similarly, we denote the unique element in $I_2 \setminus Z_i$ by $g(x_i)$. Doing this for each $i \in [1, m]$, we get two mappings $f, g : I_1 \to I_2$. To be precise, $f(x_i) = y_j$ if and only if $I_2 \setminus Y_i = \{y_j\}$, and $g(x_i) = y_j$ if and only if $I_2 \setminus Z_i = \{y_j\}$. We make some observations about the sets Y_i, Z_i and the mappings f, g in the lemmas below.

LEMMA 4.12. *For any $i \in [1, m]$, if $y, y' \in Y_i$ or $y, y' \in Z_i$, then $2y - y' \neq x_i$.*

Proof: We only provide the argument for $y, y' \in Y_i$, as the argument for $y, y' \in Z_i$ is similar. If $y, y' \in Y_i$, then there exist $x_k, x_l \in I_1$ such that $y = 2x_k - x_i$ and $y' = 2x_l - x_i$. So, if $2y - y' = x_i$, we would get $2(2x_k - x_l - x_i) = 0$, from which we obtain $2x_k - x_l = x_i$. But, this would mean that $\{x_l, x_k, x_i\}$ is a 3-term A.P. in I_1, contradicting Lemma 4.10. □

LEMMA 4.13. *f, g are bijections, and $f(x) = g(x)$ for all $x \in I_1$.*

Proof: We shall show that f is a bijection, and that $f(x) = g(x)$ for all $x \in I_1$. It is then clear that g is also a bijection. Since I_1, I_2 are finite sets of the same cardinality, to prove that f is a bijection, it suffices to show that f is injective.

Now, suppose that f is not injective. Without loss of generality, assume that $f(x_{m-1}) = f(x_m) = y_m$. By the definition of f, $\{y_m\} = I_2 \setminus Y_{m-1} = I_2 \setminus Y_m$, and hence $Y_m = Y_{m-1} = \{y_1, y_2, \ldots, y_{m-1}\}$. Now, if $y_i, y_j \in Y_m = Y_{m-1}$ are such that $y_i \neq y_j$, then by Lemma 4.12, $2y_i - y_j \notin \{x_m, x_{m-1}\}$. However, as $2y_i - y_j$ must be in I_1, we find that $2y_i - y_j \in \{x_1, x_2, \ldots, x_{m-2}\}$. This means that the sets $\{y_1, y_2, \ldots, y_{m-1}\}$ and $\{x_1, x_2, \ldots, x_{m-2}\}$ satisfy the assumptions of Lemma 4.8, and therefore, we must have $p = 3$. But this is impossible as $p \geq |I_1| + |I_2| = 2m \geq 4$. This shows that f is injective, and hence a bijection.

To show that $f(x) = g(x)$ for all $x \in I_1$, suppose to the contrary that $g(x_m) = y_m$, but $f(x_m) \neq y_m$. Now, since f is a bijection, there exists an $x \in I$, $x \neq x_m$, such that $f(x) = y_m$. Re-labeling the x_i's if necessary, we may take $f(x_{m-1}) = y_m$. We thus have $f(x_{m-1}) = g(x_m) = y_m$. Now, using the same argument as used earlier to prove the injectivity of f, except that now we replace Y_m by Z_m, we reach the conclusion via Lemma 4.8 that $p = 3$, which is impossible. Thus, we must have $f(x) = g(x)$ for all $x \in I_1$. □

We are now ready to prove the following result, which is the last step in our proof of Theorem 4.3.

Proposition 4.14. *For any $I \subset \mathbb{Z}/p$, $p \geq 3$, there does not exist a TGP-coloring of I such that $|I_1| = |I_2| \geq 3$.*

Proof: Suppose there exists such a coloring of I. Let $I_1 = \{x_1, x_2 \ldots, x_m\}$ and $I_2 = \{y_1, y_2, \ldots, y_m\}$, $m \geq 3$. Note that $\{2y_i - y_1 : i \in [2, m]\}$ lies in I_1, and all its $m-1$ elements are distinct. By re-labeling the x_j's if necessary, we may assume that $2y_i - y_1 = x_i$ for all $i \in [2, m]$.

Let $f, g : I_1 \to I_2$ be the mappings defined as above. We shall show that for any $i \in [2, m]$, $f(x_i) = y_1$. But this leads to a contradiction, since for $m \geq 3$, there exist $x_2, x_3 \in I_1$, $x_2 \neq x_3$, for which $f(x_2) = f(x_3) = y_1$, which is impossible as f is a bijection.

So, consider an arbitrary $x_i \in I_1$ with $i \in [2, m]$, and suppose that $f(x_i) \neq y_1$, Thus,

$$x_i = 2y_i - y_1 \tag{5}$$

and there exists an $x_j \in I_1$ such that

$$2x_j - x_i = y_1 \tag{6}$$

If $j \geq 2$ as well, then we would also have

$$x_j = 2y_j - y_1 \tag{7}$$

Therefore, plugging in (5) and (7) in (6), we would obtain $2(2y_j - y_i - y_1) = 0$, which would imply that $2y_j - y_i = y_1$, which is impossible by Lemma 4.10. Thus, $j = 1$, so that we must have

$$2x_1 - x_i = y_1 \tag{8}$$

Since, by Lemma 4.13, $f(x_i) = g(x_i)$, we also have $g(x_i) \neq y_1$. Now, an argument similar to the one above shows that

$$2x_i - x_1 = y_1 \tag{9}$$

But from (8) and (9), we get $2x_1 - x_i = 2x_i - x_1$, or equivalently, $3(x_i - x_1) = 0$ which is impossible as $p \geq |I_1| + |I_2| = 2m \geq 6$.

Thus, we are forced to conclude that $f(x_i) = y_1$, and since this holds for any $i \in [2, m]$, this contradicts the fact that f is a bijection. □

This concludes our proof of Theorem 4.3.

4.2. Aperiodic sequences in $\mathcal{S}_3$. We shall now work towards a proof for Theorem 4.4. It is easy to show the sufficiency of conditions (i), (ii) or (iii) in the statement of the theorem, so we proceed to do that first. For $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$ such that condition (i) holds, we construct an $\mathbf{x} = (x_j)_{j\in\mathbb{Z}} \in \mathcal{T}_3$ with $\mathrm{supp}(\mathbf{x}) = \mathrm{supp}(\mathbf{y})$ as follows: if $\mathrm{supp}(\mathbf{y}) = \{m\}$ for some $m \in \mathbb{Z}$, then simply take $\mathbf{x} = \mathbf{y}$; if $\mathrm{supp}(\mathbf{y}) = \{m, n\}$ for $m, n \in \mathbb{Z}$, $m \neq n$, then set $x_m = 1$, $x_n = 2$, and $x_j = 0$ otherwise. For $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$ such that $\mathrm{supp}(\mathbf{y}) = (k\mathbb{Z}+i) \cup V$ as in condition (ii), let $\mathbf{x} \in \{0,1,2\}^{\mathbb{Z}}$ be the sequence for which $V_1(\mathbf{x}) = k\mathbb{Z}+i$ and $V_2(\mathbf{x}) = V$. For $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$ such that $\mathrm{supp}(\mathbf{y}) = (3t\mathbb{Z}+i)\cup V$ as in condition (iii), let $\mathbf{x} \in \{0,1,2\}^{\mathbb{Z}}$ be the sequence for which $V_1(\mathbf{x}) = 3t\mathbb{Z} + i$ and $V_2(\mathbf{x}) = V$. In both these cases, it is straightforward to verify that $\mathbf{x} \in \mathcal{T}_3$, and hence, $\mathbf{y} = \pi(\mathbf{x}) \in \mathcal{S}_3$.

To prove the converse part of the theorem, we use the following approach. We first show that if $\mathbf{y} \in \mathcal{S}_3$ is such that $\overline{d}(\mathrm{supp}(\mathbf{y})) = 0$, then $|\mathrm{supp}(\mathbf{y})| \leq 2$. Thus, if $\mathbf{y} \in \mathcal{S}_3$ is aperiodic with $\overline{d}(\mathrm{supp}(\mathbf{y})) = 0$, then we must have $1 \leq |\mathrm{supp}(\mathbf{y})| \leq 2$, since $|\mathrm{supp}(\mathbf{y})| = 0$ implies that $\mathbf{y}$ is the all-zeros sequence, which is periodic. For aperiodic $\mathbf{y} \in \mathcal{S}_3$ with $\overline{d}(\mathrm{supp}(\mathbf{y})) > 0$, we analyze sequences in the set $\pi^{-1}(\mathbf{y}) \cap \mathcal{T}_3$, showing finally that there must exist a sequence $\mathbf{x} \in \pi^{-1}(\mathbf{y}) \cap \mathcal{T}_3$ such that $V_1(\mathbf{x}) = k\mathbb{Z} + i$ for some $k \in \mathbb{N}$ and $i \in [0, k-1]$, and $V_2(\mathbf{x})$ is one of the V's in conditions (ii) and (iii) in the statement of Theorem 4.4.

LEMMA 4.15. *If $\mathbf{y} \in \mathcal{S}_3$ is such that $\overline{d}(\mathrm{supp}(\mathbf{y})) = 0$, then $|\mathrm{supp}(\mathbf{y})| \leq 2$.*

Proof: Suppose that $\mathbf{y} \in \mathcal{S}_3$ is such that $|\mathrm{supp}(\mathbf{y})| \geq 3$. We shall show that $\overline{d}(\mathrm{supp}(\mathbf{y})) > 0$. Let $\mathbf{x} \in \mathcal{T}_3$ be any sequence with $\pi(\mathbf{x}) = \mathbf{y}$, so that $|V_1(\mathbf{x})| + |V_2(\mathbf{x})| = |\mathrm{supp}(\mathbf{x})| \geq 3$. Thus, either $|V_1(\mathbf{x})| \geq 2$ or $|V_2(\mathbf{x}) \geq 2$. We shall assume that $|V_1(\mathbf{x})| \geq 2$, as a symmetric argument applies to the other case. Our goal is to show that either $V_1(\mathbf{x})$ or $V_2(\mathbf{x})$ contains a 3-term A.P. $\{a, a+k, a+2k\}$ for some $a \in \mathbb{Z}$ and $k \in \mathbb{N}$. For if this is the case, then applying Corollary 3.3 with $q = 3$, noting that $S(2) = 5$, we find that $a + k\mathbb{Z} \subset \mathrm{supp}(\mathbf{x})$. Therefore, we would have $\overline{d}(\mathrm{supp}(\mathbf{y})) = \overline{d}(\mathrm{supp}(\mathbf{x})) \geq \overline{d}(a + k\mathbb{Z}) = 1/k > 0$, which proves the lemma.

Since $|V_1(\mathbf{x})| \geq 2$, pick any pair of integers $r, s \in V_1(\mathbf{x})$, $r < s$, and let $d = s - r$. Now, suppose that neither $V_1(\mathbf{x})$ or $V_2(\mathbf{x})$ contains a 3-term A.P. Since $\mathbf{x} \in \mathcal{T}_3$ and $r, r+d \in V_1(\mathbf{x})$, we must have $r + 2d \in V_1(\mathbf{x}) \cup V_2(\mathbf{x})$, as $r + 2d = (r+d) + (r+d) - r$. But as $V_1(\mathbf{x})$ does not contain a 3-term A.P., $r + 2d \in V_2(\mathbf{x})$. Similarly, $r - d \in V_2(\mathbf{x})$, as $r - d = r + r - (r + d)$. Now, applying a similar argument to the pair of integers $r - d, r + 2d \in V_2(\mathbf{x})$, we find that $r - 4d, r + 5d \in V_1(\mathbf{x})$. Next, $r + d, r + 5d \in V_1(\mathbf{x})$ implies that $r - 3d \in V_2(\mathbf{x})$. Finally, since $r - d, r - 3d \in V_2(\mathbf{x})$, we have $r - 5d \in V_1(\mathbf{x})$. But now, $\{r - 5d, r, r + 5d\}$ is a 3-term A.P. in $V_1(\mathbf{x})$, contradicting our assumption. Hence, if $|V_1(\mathbf{x})| \geq 2$, then either $V_1(\mathbf{x})$ or $V_2(\mathbf{x})$ contains a 3-term A.P., thus proving the lemma. ☐

As explained earlier, the above lemma shows that if $\mathbf{y} \in \mathcal{S}_3$ is aperiodic with $\overline{d}(\mathrm{supp}(\mathbf{y})) = 0$, then $1 \leq \mathrm{supp}(\mathbf{y}) \leq 2$, which is condition (i) of Theorem 4.4.

So now, we are left to deal with the set of aperiodic sequences $\mathbf{y} \in \mathcal{S}_3$ with $\overline{d}(\mathrm{supp}(\mathbf{y})) > 0$, which we shall denote by $\mathcal{Q}_3$. As before, for $\mathbf{x} \in \mathcal{T}_3$, we define $P(\mathbf{x}) = \bigcup_{j\in J_1} V_j(\mathbf{x})$, where $J_1 = \{j \in \{1,2\} : \overline{d}(V_j(\mathbf{x})) > 0\}$, and $N(\mathbf{x}) = \bigcup_{j\in J_2} V_j(\mathbf{x})$, where $J_2 = \{j \in \{1,2\} : \overline{d}(V_j(\mathbf{x})) = 0\}$. Note that if $\mathbf{x} \in \mathcal{T}_3$ is such that $\pi(\mathbf{x}) \in \mathcal{Q}_3$, then $P(\mathbf{x}) \neq \emptyset$, since $\overline{d}(\mathrm{supp}(\mathbf{x})) = \overline{d}(\mathrm{supp}(\pi(\mathbf{x}))) > 0$, and by Corollary 3.5, $N(\mathbf{x}) \neq \emptyset$ as well, since $\pi(\mathbf{x})$ is aperiodic. Thus, for any $\mathbf{x} \in \mathcal{T}_3$ such that $\pi(\mathbf{x}) \in \mathcal{Q}_3$, we have

$\{P(\mathbf{x}), N(\mathbf{x})\} = \{V_1(\mathbf{x}), V_2(\mathbf{x})\}$.

The proof of the remaining part of Theorem 4.4 begins with the following lemma.

LEMMA 4.16. *Let* $\mathbf{x} \in \mathcal{T}_3$ *be such that* $\pi(\mathbf{x}) \in \mathcal{Q}_3$. *If there exists a* $k \in \mathbb{N}$ *and an* $I \subset [0, k-1]$ *such that* $P(\mathbf{x}) \subset \bigcup_{i\in I}(k\mathbb{Z}+i) \subset \mathrm{supp}(\mathbf{x})$, *then the elements of* $\mathrm{supp}(\mathbf{x}) \setminus \bigcup_{i\in I}(k\mathbb{Z}+i)$ *are all distinct modulo* k. *Hence,* $|\mathrm{supp}(\mathbf{x}) \setminus \bigcup_{i\in I}(k\mathbb{Z}+i)| \le k$.

Proof: Let $N'(\mathbf{x}) = \mathrm{supp}(\mathbf{x}) \setminus \bigcup_{i\in I}(k\mathbb{Z}+i)$. Note that under the assumptions of the lemma, $N'(\mathbf{x}) \subset \mathrm{supp}(\mathbf{x}) \setminus P(\mathbf{x}) = N(\mathbf{x})$. We shall show that if $N'(\mathbf{x})$ contains distinct integers a, b such that $a \equiv b \pmod{k}$, then $\overline{d}(N'(\mathbf{x})) > 0$. This leads to the contradiction that $\overline{d}(N(\mathbf{x})) > 0$, since $\overline{d}(N(\mathbf{x})) \ge \overline{d}(N'(\mathbf{x}))$.

So, suppose that $a, b \in N'(\mathbf{x})$, $a < b$, are such that $a \equiv b \pmod{k}$. For any $j \in \mathrm{supp}(\mathbf{x})$, by the definition of $N'(\mathbf{x})$, $j \in N'(\mathbf{x})$ if and only if $j \mod k \notin I$. In particular, $a \equiv b \equiv \ell \pmod{k}$ for some $\ell \notin I$. Let $d = b - a$, and note that $d \equiv 0 \pmod{k}$.

As observed above, for any $\mathbf{x} \in \mathcal{T}_3$ such that $\pi(\mathbf{x}) \in \mathcal{Q}_3$, we have $\{P(\mathbf{x}), N(\mathbf{x})\} = \{V_1(\mathbf{x}), V_2(\mathbf{x})\}$. Without loss of generality, we may assume that $P(\mathbf{x}) = V_1(\mathbf{x})$ and $N(\mathbf{x}) = V_2(\mathbf{x})$, and hence, $N'(\mathbf{x}) \subset V_2(\mathbf{x})$. So, we have $a, a+d \in V_2(\mathbf{x})$, and hence by the TGP condition, $a + 2d \in \mathrm{supp}(\mathbf{x})$. But since $a + 2d \equiv \ell \pmod{k}$, and $\ell \notin I$, we must have $a + 2d \in N'(\mathbf{x}) \subset V_2(\mathbf{x})$. But now, $V_2(\mathbf{x})$ contains the 3-term A.P. $\{a, a+d, a+2d\}$, so that by Corollary 3.3, $a + d\mathbb{Z} \subset \mathrm{supp}(\mathbf{x})$. However, for any $j \in a + d\mathbb{Z}$, $j \equiv \ell \pmod{k}$, and hence $j \in N'(\mathbf{x})$. Thus, $a + d\mathbb{Z} \subset N'(\mathbf{x})$, from which we obtain $\overline{d}(N'(\mathbf{x})) \ge \overline{d}(a + d\mathbb{Z}) = 1/d > 0$, which leads to the contradiction that proves the lemma. □

Lemma 4.16 leads us to the following result, which is the crucial step in our proof of the rest of the converse part of Theorem 4.4.

LEMMA 4.17. *For each* $\mathbf{y} \in \mathcal{Q}_3$, *there exists an* $\mathbf{x} \in \mathcal{T}_3$ *such that* $\pi(\mathbf{x}) = \mathbf{y}$ *and* $P(\mathbf{x}) = \bigcup_{i\in I}(k\mathbb{Z}+i)$ *for some* $k \in \mathbb{N}$ *and* $I \subset [0, k-1]$.

Proof: Consider an arbitrary $\mathbf{y} \in \mathcal{Q}_3$. Since $\mathcal{Q}_3 \subset \mathcal{S}_3$, there exists a $\mathbf{z} \in \mathcal{T}_3$ such that $\pi(\mathbf{z}) = \mathbf{y}$. As observed prior to Lemma 4.16, for such a $\mathbf{z}$, we have $\{P(\mathbf{z}), N(\mathbf{z})\} = \{V_1(\mathbf{z}), V_2(\mathbf{z})\}$. Without loss of generality, we may assume that $P(\mathbf{z}) = V_1(\mathbf{z})$ and $N(\mathbf{z}) = V_2(\mathbf{z})$. Since $\overline{d}(V_1(\mathbf{z})) = \overline{d}(P(\mathbf{z})) > 0$, by Lemma 3.4, there exists a $k \in \mathbb{N}$ and an $I \subset [0, k-1]$ such that $V_1(\mathbf{z}) \subset \bigcup_{i\in I}(k\mathbb{Z}+i) \subset \mathrm{supp}(\mathbf{z})$. Moreover, by Lemma 4.16, $\mathrm{supp}(\mathbf{z}) \setminus \bigcup_{i\in I}(k\mathbb{Z}+i)$ is a finite set.

Note that for any $i \in I$, $k\mathbb{Z}+i \subset \mathrm{supp}(\mathbf{z}) = V_1(\mathbf{z}) \cup V_2(\mathbf{z})$, so that $\overline{d}(k\mathbb{Z}+i) = \overline{d}(V_1(\mathbf{z}) \cap (k\mathbb{Z}+i)) + \overline{d}(V_2(\mathbf{z}) \cap (k\mathbb{Z}+i))$. Since $\overline{d}(V_2(\mathbf{z}) \cap (k\mathbb{Z}+i)) = 0$, we have $\overline{d}(V_1(\mathbf{z}) \cap (k\mathbb{Z}+i)) = \overline{d}(k\mathbb{Z}+i) = 1/k > 0$. In particular, $V_1(\mathbf{z}) \cap (k\mathbb{Z}+i)$ is an infinite set for any $i \in I$.

Now, let $\mathbf{x} = (x_j)_{j\in\mathbb{Z}} \in \{0,1,2\}^{\mathbb{Z}}$ be defined as follows:

$$x_j = \begin{cases} 0 & \forall\, j \notin \mathrm{supp}(\mathbf{z}) \\ 1 & \forall\, j \in \bigcup_{i\in I}(k\mathbb{Z}+i) \\ 2 & \forall\, j \in \mathrm{supp}(\mathbf{z}) \setminus \bigcup_{i\in I}(k\mathbb{Z}+i) \end{cases}$$

It is clear that $\pi(\mathbf{x}) = \mathbf{y}$, since $\mathrm{supp}(\mathbf{x}) = \mathrm{supp}(\mathbf{z}) = \mathrm{supp}(\mathbf{y})$. Also, we have $\overline{d}(V_1(\mathbf{x})) \ge 1/k > 0$, and since $V_2(\mathbf{x})$ is a finite set, $\overline{d}(V_2(\mathbf{x})) = 0$. Hence, $P(\mathbf{x}) =$

$V_1(\mathbf{x}) = \bigcup_{i\in I}(k\mathbb{Z}+i)$. It only remains to show that $\mathbf{x} \in \mathcal{T}_3$, *i.e.*, to show that if $p,q,r \in V_1(\mathbf{x})$ or $p,q,r \in V_2(\mathbf{x})$, then $p+q-r \in \operatorname{supp}(\mathbf{x})$.

Note that $V_2(\mathbf{x}) \subset V_2(\mathbf{z})$. Therefore, if we take any $p,q,r \in V_2(\mathbf{x})$, then $p,q,r \in V_2(\mathbf{z})$ as well, and hence, since $\mathbf{z} \in \mathcal{T}_3$, $p+q-r \in \operatorname{supp}(\mathbf{z}) = \operatorname{supp}(\mathbf{x})$.

Now, let $p,q,r \in V_1(\mathbf{x})$, and suppose that $p+q-r \notin \operatorname{supp}(\mathbf{x})$. Thus, $p+q-r \equiv j \pmod k$ for some $j \notin I$. As shown above, for any $i \in I$, $V_1(\mathbf{z}) \cap (k\mathbb{Z}+i)$ is an infinite set. Hence, we can pick $q', r' \in V_1(\mathbf{z})$ such that $q' \equiv q \pmod k$ and $r' \equiv r \pmod k$, and furthermore, we can pick a $p' \in V_1(\mathbf{z})$, with $p' \equiv p \pmod k$, that is large enough in absolute value that $p'+q'-r'$ lies outside the finite set $\operatorname{supp}(\mathbf{z}) \setminus \bigcup_{i\in I}(k\mathbb{Z}+i)$. Thus, $p'+q'-r' \notin \operatorname{supp}(\mathbf{z}) \setminus \bigcup_{i\in I}(k\mathbb{Z}+i)$, and as $p'+q'-r' \equiv p+q-r \equiv j \pmod k$, we see that $p'+q'-r' \notin \bigcup_{i\in I}(k\mathbb{Z}+i)$ either. This shows that $p'+q'-r' \notin \operatorname{supp}(\mathbf{z})$, which contradicts the fact that $\mathbf{z} \in \mathcal{T}_3$, Hence, we must have $p+q-r \in \operatorname{supp}(\mathbf{x})$, which shows that $\mathbf{x} \in \mathcal{T}_3$, thus proving the result. $\square$

In the next two lemmas, we show that the sequence $\mathbf{x} \in \mathcal{T}_3$ whose existence is guaranteed by Lemma 4.17, must in fact have $P(\mathbf{x}) = k_0\mathbb{Z}+i_0$ for some $k_0 \in \mathbb{N}$ and $i_0 \in [0, k_0-1]$, and $N(\mathbf{x}) = V$, where V is as in condition (ii) or (iii) of the theorem.

Lemma 4.18. *Let $\mathbf{x} \in \mathcal{T}_3$ be such that $\pi(\mathbf{x}) \in \mathcal{Q}_3$. If there exists a $k \in \mathbb{N}$ and an $I \subset [0,k-1]$ such that $P(\mathbf{x}) = \bigcup_{i\in I}(k\mathbb{Z}+i)$, then $\bigcup_{i\in I}(k\mathbb{Z}+i) = d\mathbb{Z}+\ell$ for some $d \in \mathbb{N}$ such that $d|k$, and some $\ell \in [0,d-1]$.*

Proof: Our goal is to show that under the assumptions of the lemma, I must be a coset of some subgroup of the group, $\mathbb{Z}/k$, of integers modulo k. (We represent $\mathbb{Z}/k$ here as the set $[0,k-1]$ equipped with the operation of modulo-k addition.) Since any (non-empty) subgroup of $\mathbb{Z}/k$ is generated by some divisor d of k, any such I must be of the form $\{i \in [0,k-1] : \ i \equiv \ell \pmod d\}$ for some $d \in \mathbb{N}$, $d|k$, and some $\ell \in [0,d-1]$. It then immediately follows that $\bigcup_{i\in I}(k\mathbb{Z}+i) = d\mathbb{Z}+\ell$, as stated in the lemma.

Now, to show that I is a coset of some subgroup of $\mathbb{Z}/k$, it is enough to show that I is closed under the ternary operation $i_1+i_2-i_3 \mod k$, *i.e.*, if $i_1,i_2,i_3 \in I$, then $i_1+i_2-i_3 \mod k \in I$. Indeed, if I is closed under this operation, then take any $\ell \in I$ and consider the set $H = \{i-\ell \mod k : i \in I\}$. It is easily verified that H is a subgroup of $\mathbb{Z}/k$, and so I is a coset of H.

Thus, it only remains to prove that under the assumptions of the lemma, if $i_1,i_2,i_3 \in I$, then $i_1+i_2-i_3 \mod k \in I$. As $\mathbf{x} \in \mathcal{T}_3$ is such that $\pi(\mathbf{x}) \in \mathcal{Q}_3$, we may assume that $P(\mathbf{x}) = V_1(\mathbf{x})$ and $N(\mathbf{x}) = V_2(\mathbf{x})$. Thus, $V_1(\mathbf{x}) = \bigcup_{i\in I}(k\mathbb{Z}+i)$ and $V_2(\mathbf{x}) = \operatorname{supp}(\mathbf{x}) \setminus \bigcup_{i\in I}(k\mathbb{Z}+i)$. Note that k and I satisfy the assumptions of Lemma 4.16, and hence, we have $|V_2(\mathbf{x})| = |\operatorname{supp}(\mathbf{x}) \setminus \bigcup_{i\in I}(k\mathbb{Z}+i)| \le k$. In other words, $V_2(\mathbf{x})$ is a finite set.

Now, consider any $i_1,i_2,i_3 \in I$. Since $V_2(\mathbf{x})$ is a finite set, we can choose an integer r large enough that $kr+(i_1+i_2-i_3) \notin V_2(\mathbf{x})$. Let $r_1,r_2,r_3 \in \mathbb{Z}$ be such that $r_1+r_2-r_3 = r$. Note that for $j = 1,2,3$, $kr_j+i_j \in V_1(\mathbf{x})$, since $k\mathbb{Z}+i_j \subset V_1(\mathbf{x})$. Hence, by the TGP condition applied to $kr_1+i_1, kr_2+i_2, kr_3+i_3 \in V_1(\mathbf{x})$, we obtain $kr+(i_1+i_2-i_3) \in \operatorname{supp}(\mathbf{x})$. Since $kr+(i_1+i_2-i_3)$ is not in $V_2(\mathbf{x})$, it must be in $V_1(\mathbf{x}) = \bigcup_{i\in I}(k\mathbb{Z}+i)$, from which it follows that $i_1+i_2-i_3 \mod k \in I$, as desired. $\square$

Lemma 4.19. *Let $\mathbf{x} \in \mathcal{T}_3$ be such that $\pi(\mathbf{x}) \in \mathcal{Q}_3$, and $P(\mathbf{x}) = k\mathbb{Z}+i$ for some $k \in \mathbb{N}$ and $i \in [0,k-1]$. Then, $1 \le |N(\mathbf{x})| \le 2$. Furthermore, if $|N(\mathbf{x})| = 2$, then*

$k = 3t$ for some $t \in \mathbb{N}$ and $N(\mathbf{x}) \equiv \{t+i, 2t+i\} \pmod{3t}$.

Proof: Since $\mathbf{x}$ satisfies the assumptions of Lemma 4.16, and $N(\mathbf{x}) = \text{supp}(\mathbf{x}) \setminus (k\mathbb{Z}+i)$, we find that $|N(\mathbf{x})| \leq k$. Furthermore, if a, b are distinct integers in $N(\mathbf{x})$, then $a \not\equiv b \pmod{k}$. As usual, we shall assume that $P(\mathbf{x}) = V_1(\mathbf{x})$ and $N(\mathbf{x}) = V_2(\mathbf{x})$, so that $V_1(\mathbf{x}) = k\mathbb{Z}+i$ and $V_2(\mathbf{x})$ is a finite set. Furthermore, since $\pi(\mathbf{x})$ is aperiodic, $V_2(\mathbf{x})$ cannot be empty, *i.e.*, $|V_2(\mathbf{x})| \geq 1$. We shall show that if $|V_2(\mathbf{x})| > 1$, then we must have $|V_2(\mathbf{x})| = 2$, $k = 3t$ for some $t \in \mathbb{N}$, and $V_2(\mathbf{x}) \equiv \{t+i, 2t+i\} \pmod{3t}$, which would prove the lemma.

So, suppose that $|V_2(\mathbf{x})| \geq 2$. Let a be the smallest integer in $V_2(\mathbf{x})$, and b be the largest, so that $a < b$. Note that since a and b are distinct integers in $V_2(\mathbf{x}) = N(\mathbf{x})$, we have $a \not\equiv b \pmod{k}$. Now, applying the TGP condition to $a, b \in V_2(\mathbf{x})$, we find that $2a-b, 2b-a \in \text{supp}(\mathbf{x})$. However, $2a-b < a$, and since a is the smallest integer in $V_2(\mathbf{x})$, $2a-b$ cannot be in $V_2(\mathbf{x})$. Hence, $2a-b \in V_1(\mathbf{x})$. A similar argument shows that $2b-a \in V_1(\mathbf{x})$ as well. But since $V_1(\mathbf{x}) = k\mathbb{Z}+i$, we have

$$2a - b \equiv 2b - a \equiv i \pmod{k} \tag{10}$$

Therefore, $3(a-b) \equiv 0 \pmod{k}$. Since $a \not\equiv b \pmod{k}$, 3 must divide k and $a \equiv b \pmod{k/3}$.

Thus, $k = 3t$ for some $t \in \mathbb{N}$, and so we have $a \equiv b \pmod{t}$, but $a \not\equiv b \pmod{3t}$. Therefore, either $b \equiv t + a \pmod{3t}$ or $b \equiv 2t + a \pmod{3t}$. But since a, b must also satisfy the congruence $2b - a \equiv i \pmod{3t}$ in (10), some simple manipulations now show that $\{a, b\} \equiv \{t+i, 2t+i\} \pmod{3t}$.

Now, if there exists a $c \in V_2(\mathbf{x})$ such that $a < c < b$, then a similar argument as used for (10) establishes that $a + c - b \equiv b + c - a \equiv i \pmod{k}$, and hence

$$2(a-b) \equiv 0 \pmod{k} \tag{11}$$

Using $k = 3t$ and $\{a, b\} \equiv \{t+i, 2t+i\} \pmod{3t}$, it follows from (11) that either $t \equiv 0 \pmod{3t}$ or $2t \equiv 0 \pmod{3t}$, both of which are impossible for $t \neq 0$. Hence, if $|V_2(\mathbf{x})| > 1$, then $V_2(\mathbf{x})$ cannot contain anything other than the two integers a, b as above, which completes the proof of the lemma. $\square$

From Lemmas 4.17, 4.18 and 4.19, we see that for any $\mathbf{y} \in \mathcal{Q}_3$, either $\text{supp}(\mathbf{y})$ is of the form given in condition (ii), or it must be as in condition (iii) of Theorem 4.4, which completes the proof of that theorem.

5. Numerical Results and Conjectures. For $k \in \mathbb{N}$, let $P(k)$ denote the number of TGP-colorable subsets of $\mathbb{Z}/k$. It follows from Theorem 4.3 that $P(2) = 4$, $P(3) = 8$, $P(5) = 22$, and for primes $p > 5$, $P(p) = 1 + p + \binom{p}{2} + 1 = p(p+1)/2 + 2$. However, for non-prime k, we do not have a simple means of computing $P(k)$, as we do not have a complete solution to the problem of determining precisely which subsets of $\mathbb{Z}/k$ are TGP-colorable. We list the values of $P(k)$ for non-prime $k \leq 20$ in Table 5.1 below, most of which have been obtained by means of an exhaustive computer search.

The above table seems to suggest that $P(k)$ grows slowly, perhaps polynomially, with k. Now, recall our definition of $\mathcal{B}_{3,n}$ as the set of all n-blocks of $\mathcal{S}_3$. It can be inferred from Theorem 4.4 that aperiodic sequences in $\mathcal{S}_3$ contribute $O(n^4)$ blocks to $\mathcal{B}_{3,n}$. Based on the slow growth rate of $P(k)$, we conjecture that the number of blocks

k	1	4	6	8	9	10	12	14	15	16	18	20
$P(k)$	2	16	52	80	98	134	340	228	328	384	808	746

TABLE 5.1
The number, $P(k)$, of subsets of $\mathbb{Z}/k$ that are TGP-colorable, for non-prime $k \leq 20$.

contributed to $\mathcal{B}_{3,n}$ by periodic sequences in $\mathcal{S}_3$ is also polynomial in n. Consequently, we conjecture that $h(\mathcal{S}_3) = 0$, just as in the BGP case.

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