A COMPARATIVE STUDY OF PERIODS IN A PERIODIC-FINITE-TYPE SHIFT*

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Abstract. Periodic-finite-type shifts (PFT's) form a class of sofic shifts that strictly contains the class of shifts of finite type (SFT's). In this paper, we study PFT's from the viewpoint of certain "periods" that can be associated with them. We define three kinds of periods (descriptive, sequential and graphical) for PFT's, and investigate the relationships between them. The results of our investigation indicate that there are no specific relationships between these periods, except for the fact that the descriptive period of an irreducible PFT always divides its graphical period. Furthermore, we compute the number of periodic sequences in PFT's of a certain type, from which we obtain expressions for their zeta functions.

Key words. periodic-finite-type shifts, shifts of finite type, zeta functions

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1. Introduction. A shift of finite type (SFT) is defined as a set of bi-infinite sequences (over some alphabet) that do not contain as subwords any word from a certain finite set. SFT's are objects of fundamental importance in symbolic dynamics [4] and the theory of constrained coding [5].

A generalization of SFT's was introduced by Moision and Siegel [7] who were interested in examining the properties of distance-enhancing constrained codes, in which the appearance of certain words is forbidden in a periodic manner. This new class of shifts, called periodic-finite-type shifts (PFT's), contains the class of SFT's and some other interesting classes of shifts, such as constrained systems with unconstrained positions [1],[9], and shifts arising from the time-varying maximum transition run constraint [8]. The class of PFT's is in turn properly contained within the class of sofic shifts [6], a fact we discuss in more detail in Section 2.

The difference between the definitions of SFT's and PFT's is small, but significant. An SFT is defined by forbidding the appearance of finitely many words at any position of a bi-infinite sequence. A PFT is also defined by forbidding the appearance of finitely many words within a bi-infinite sequence, except that these words are only forbidden to appear at positions indexed by certain pre-defined *periodic* integer sequences; see Section 2 for a formal definition. Thus, there is a notion of period inherent in the definition of a PFT that causes it to differ from an SFT.

The properties of SFT's are quite well understood (see, for example, [4]), but the same cannot be said for PFT's. The study of PFT's has, up to this point, primarily focused on finding efficient algorithms for constructing their presentations [1],[2],[6]. The work presented in this paper began as an attempt to extend some of what is known about SFT's to the larger class of PFT's. In particular, we wanted to see whether we could come up with a simple formula for computing the zeta function of a PFT, analogous to the one known for an SFT [4, Theorem 6.4.6].

Recall that the zeta function of a shift $\mathcal{S}$ is a generating function for the number of sequences of period n in $\mathcal{S}$; see Section 5 for a precise definition. It is known that

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the zeta function of a sofic shift $\mathcal{S}$ is a rational function, an expression for which can be found in Theorem 6.4.8 of [4]. As mentioned above, PFT's are indeed sofic shifts, so, in principle, there is a method known to compute their zeta functions. However, the method of Theorem 6.4.8 in [4] quickly becomes too cumbersome for practical computations. So we made an attempt to determine whether it was possible to simplify such computations in the special case of PFT's. We are as yet unable to resolve this question. However, in our pursuit of zeta functions of PFT's, we found ourselves asking the question of whether the notion of period inherent in the definition of a PFT affects the periods of periodic sequences in the PFT. It is this question, along with some other closely related ones, that we address in this paper.

We focus on three types of period that can be associated with a PFT $\mathcal{X}$: (i) the *descriptive* period in the definition of the PFT; (ii) the *sequential* period, which we take to be the least period of any periodic sequence in $\mathcal{X}$; and (iii) the *graphical* period (only defined for irreducible PFT's), which is the least period of any irreducible presentation of $\mathcal{X}$. Formal definitions of these periods can be found in Section 4, where we investigate the relationships that exist between them. Our investigations indicate that there are no simple relationships between these periods in general, except for one: the descriptive period of an irreducible PFT must always divide its graphical period. This last fact actually implies a result of Moision and Siegel [6, Proposition 1].

As part of our comparative study of periods in PFT's, we give various examples that illustrate some of the phenomena involved in the interplay of periods. For instance, we construct a class of PFT's whose descriptive periods can be arbitrarily large compared to their sequential periods. We also give a class of PFT's where the opposite phenomenon occurs. We give similar examples for other pairs of period types as well (except, of course, there are no examples in which the descriptive period is larger than the graphical period, since the former always divides the latter).

The class of PFT's that we use to illustrate the fact that sequential periods can be much larger than descriptive periods has another remarkable property. Each PFT $\mathcal{X}_k$ in this class is parametrized by a positive integer k, and periodic sequences in $\mathcal{X}_k$ can only have periods that are multiples of $2^{\lceil \log_2 k \rceil}$. This relative paucity of periodic sequences in $\mathcal{X}_k$ allows us to compute an exact expression for its zeta function. The zeta functions $\zeta_{\mathcal{X}_k}(t)$ could serve as non-trivial test cases for validating a future general formula for the zeta function of a PFT.

The rest of this paper is organized as follows. A review of the relevant definitions and background is provided in Section 2. In Section 3, we derive a useful sufficient condition for checking whether a given PFT is irreducible (so that its graphical period can be defined). This result is used multiple times in subsequent sections. Section 4 contains our main results — Theorems 4.12, 4.14 and 4.20, and Proposition 4.18 — concerning periods in PFT's. The computation of the zeta function of the PFT's $\mathcal{X}_k$ is presented in Section 5. The proof of a relatively minor observation about the irreducibility of the PFT's $\mathcal{X}_k$ is given in an appendix.

2. Basic Background on SFT's and PFT's. We begin with a review of basic background, based on material from [4] and [6]. Let Σ be a finite set of symbols; we call Σ an *alphabet*. We always assume that $|\Sigma| = q \geq 2$ since $q = 1$ gives us a trivial case. Let $\mathbf{w} = \dots w_{-1} w_0 w_1 \dots$ be a bi-infinite sequence over Σ. A word (finite-length sequence) u with length $|u| = n$ (for some integer n) is said to be a *subword* of $\mathbf{w}$, denoted by $u \prec \mathbf{w}$, if $u = w_i w_{i+1} \dots w_{i+n-1}$ for some integer i. If we want to emphasize the fact that u is a subword of $\mathbf{w}$ starting at the index i, (*i.e.*,

$u = w_i w_{i+1} \ldots w_{i+n-1}$), we write $u \prec_i \mathbf{w}$. By convention, we assume that the empty word $\epsilon \in \Sigma^0$ is a subword of any bi-infinite sequence. The notion of subwords of bi-infinite sequences can be naturally extended to the notion of subwords of words, and we use the same notations $u \prec v$ and $u \prec_i v$ to represent that u is a subword of a word v. Also, we define σ to be the shift map, that is, $\sigma(\mathbf{w}) = \ldots w^*_{-1} w^*_0 w^*_1 \ldots$ is the bi-infinite sequence satisfying $w^*_i = w_{i+1}$ for all i.

Given a labeled directed graph $\mathcal{G}$, whose (edge) labels come from Σ, let $S(\mathcal{G})$ be the set of bi-infinite sequences which are generated by reading off labels along bi-infinite paths in $\mathcal{G}$. A *sofic shift* $\mathcal{S}$ is a set of bi-infinite sequences such that $\mathcal{S} = S(\mathcal{G})$ for some labeled directed graph $\mathcal{G}$. In this case, we say that $\mathcal{S}$ is *presented by* $\mathcal{G}$, or that $\mathcal{G}$ is a *presentation* of $\mathcal{S}$. It is well known that every sofic shift has a *deterministic* presentation, *i.e.*, a presentation such that outgoing edges from the same state (vertex) are labeled distinctly. For a sofic shift $\mathcal{S}$, $\mathcal{B}_n(\mathcal{S})$ denotes the set of words $u \in \Sigma^n$ satisfying $u \prec \mathbf{w}$ for some bi-infinite sequence $\mathbf{w}$ in $\mathcal{S}$, and $\mathcal{B}(\mathcal{S}) = \cup_{n \geq 0} \mathcal{B}_n(\mathcal{S})$. A sofic shift $\mathcal{S}$ is *irreducible* if there is an irreducible (*i.e.*, strongly connected) presentation of $\mathcal{S}$, or equivalently, for every ordered pair of words u and v in $\mathcal{B}(\mathcal{S})$, there exists a word $z \in \mathcal{B}(\mathcal{S})$ such that $uzv \in \mathcal{B}(\mathcal{S})$.

A *shift of finite type* (*SFT*) $\mathcal{Y}_{\mathcal{F}'}$, with a finite set of forbidden words (a forbidden set) $\mathcal{F}'$, is the set of all bi-infinite sequences $\mathbf{w} = \cdots w_{-1} w_0 w_1 \cdots$ over Σ such that $\mathbf{w}$ contains no word $f' \in \mathcal{F}'$ as a subword. That is, the finite number of words f' in $\mathcal{F}'$ are not in $\mathcal{B}(\mathcal{Y}_{\mathcal{F}'})$.

A *periodic-finite-type shift*, which we abbreviate as *PFT*, is characterized by an ordered list of finite sets $\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)})$ and a *period* T. The PFT $\mathcal{X}_{\{\mathcal{F},T\}}$ is defined as the set of all bi-infinite sequences $\mathbf{w}$ over Σ such that for some integer $r \in \{0, 1, \ldots, T-1\}$, the r-shifted sequence $\sigma^r(\mathbf{w})$ of $\mathbf{w}$ satisfies $u \prec_i \sigma^r(\mathbf{w}) \implies u \notin \mathcal{F}^{(i \bmod T)}$ for every integer i. For simplicity, we say that a word f is in $\mathcal{F}$ (symbolically, $f \in \mathcal{F}$) if $f \in \mathcal{F}^{(j)}$ for some j. Since the appearance of words $f \in \mathcal{F}$ is forbidden in a periodic manner, note that f can be in $\mathcal{B}(\mathcal{X}_{\{\mathcal{F},T\}})$. Also, observe that a PFT $\mathcal{X}_{\{\mathcal{F},T\}}$ satisfying $\mathcal{F}^{(0)} = \mathcal{F}^{(1)} = \cdots = \mathcal{F}^{(T-1)}$ is simply the SFT $\mathcal{Y}_{\mathcal{F}'}$ with $\mathcal{F}' = \mathcal{F}^{(0)}$. Thus, SFT's are special cases of PFT's. We call a PFT *proper* when it cannot be represented as an SFT.

Any SFT can be considered to be an SFT in which every forbidden word has the same length. More precisely, given an SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}^*}$, find the longest forbidden word in $\mathcal{F}^*$ and say it has length ℓ. Set $\mathcal{F}' = \{f' \in \Sigma^\ell : f' \text{ has some } f^* \in \mathcal{F}^* \text{ as a prefix}\}$. Then, $\mathcal{Y}_{\mathcal{F}^*} = \mathcal{Y}_{\mathcal{F}'}$, and each word in $\mathcal{F}'$ has the same length, ℓ. Furthermore, we can also assume that $\mathcal{B}_\ell(\mathcal{Y}) = \Sigma^\ell \setminus \mathcal{F}'$ since if not (that is, if $\mathcal{B}_\ell(\mathcal{Y}) \subsetneq \Sigma^\ell \setminus \mathcal{F}'$), every word in $(\Sigma^\ell \setminus \mathcal{F}') \setminus \mathcal{B}_\ell(\mathcal{Y})$ can be added to $\mathcal{F}'$, and $\mathcal{Y}$ itself remains the same.

Correspondingly, every PFT $\mathcal{X}$ has a representation of the form $\mathcal{X}_{\{\mathcal{F},T\}}$ such that $\mathcal{F}^{(j)} = \emptyset$ for $1 \leq j \leq T-1$, and every word in $\mathcal{F}^{(0)}$ has the same length. An arbitrary representation $\mathcal{X}_{\{\mathcal{F},T\}}$ can be converted to one in the above form as follows. If $f \in \mathcal{F}^{(j)}$ for some $1 \leq j \leq T-1$, list out all words with length $j + |f|$ whose suffix is f, add them to $\mathcal{F}^{(0)}$, and delete f from $\mathcal{F}^{(j)}$. Continue this process until $\mathcal{F}^{(1)} = \cdots = \mathcal{F}^{(T-1)} = \emptyset$. Then, apply the method described above for SFT's to make every word in $\mathcal{F}^{(0)}$ have the same length.

It is known that PFT's belong to the class of sofic shifts.

Theorem 2.1 (Moision and Siegel, [6]). *All periodic-finite-type shifts $\mathcal{X}$ are sofic shifts. That is, for any PFT $\mathcal{X}$, there is a presentation $\mathcal{G}$ of $\mathcal{X}$.*

Moision and Siegel proved the theorem by giving an algorithm that, given a PFT $\mathcal{X}$, generates a presentation, $\mathcal{G}_{\mathcal{X}}$, of $\mathcal{X}$. We call the presentation $\mathcal{G}_{\mathcal{X}}$ the *MS presentation* of $\mathcal{X}$. The *MS algorithm*, given a PFT $\mathcal{X}$ as input, runs as follows.

Step 1. Represent $\mathcal{X}$ in the form $\mathcal{X}_{\{\mathcal{F},T\}}$, such that every word in $\mathcal{F}$ has the same length ℓ and belongs to $\mathcal{F}^{(0)}$.
Step 2. Prepare T copies of Σ^ℓ and name them $\mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(T-1)}$.
Step 3. Consider the words in $\mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(T-1)}$ as states. Draw an edge labeled $a \in \Sigma$ from $u = u_1u_2\cdots u_\ell \in \mathcal{V}^{(j)}$ to $v = v_1v_2\cdots v_\ell \in \mathcal{V}^{(j+1 \mod T)}$ if and only if $u_2\cdots u_\ell = v_1\cdots v_{\ell-1}$ and $v_\ell = a$.
Step 4. Remove states corresponding to words in $\mathcal{F}^{(0)}$ from $\mathcal{V}^{(0)}$, together with their incoming and outgoing edges. Call this labeled directed graph $\mathcal{G}'$.
Step 5. If there is a state in $\mathcal{G}'$ having only incoming edges or only outgoing edges, remove the state from $\mathcal{G}'$ as well as its incoming or outgoing edges. Continue this process until we cannot find such a state. The resulting graph $\mathcal{G}_{\mathcal{X}}$ is a presentation of $\mathcal{X}$.

REMARK 2.1. *It is evident that the MS presentation of a PFT is always deterministic. Also, for a path α in $\mathcal{G}_{\mathcal{X}}$ with length $|\alpha| \geq \ell$, α terminates at some state that is a copy of $u = u_1u_2\ldots u_\ell$ iff the length-ℓ suffix of the word generated by α is equal to u.*

3. Irreducibility of PFT's. Recall from the previous section that a PFT $\mathcal{X}$ with period T always has a representation $\mathcal{X}_{\{\mathcal{F},T\}}$ in which

$$\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)}) = (\mathcal{F}^{(0)}, \emptyset, \ldots, \emptyset),$$

and every word in $\mathcal{F}^{(0)}$ has the same length. Thus, the following result can often be a useful means of verifying the irreducibility of a PFT. We will make repeated use of this result in the next section.

THEOREM 3.1. *Let $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ be a PFT with*

$$\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)}) = (\mathcal{F}', \emptyset, \ldots, \emptyset)$$

for some $\mathcal{F}' \subseteq \Sigma^\ell$, $\ell \geq 1$. Suppose that the SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ has the following properties:
(i) $\mathcal{Y}$ is irreducible;
(ii) $\mathcal{B}_\ell(\mathcal{Y}) = \Sigma^\ell \setminus \mathcal{F}'$; and
(iii) there exists a periodic bi-infinite sequence $\mathbf{y}$ in $\mathcal{Y}$ with a period p satisfying $p \equiv 1 \pmod T$.
Then, the MS presentation, $\mathcal{G}_{\mathcal{X}}$, of $\mathcal{X}$ is irreducible as a graph, and hence, $\mathcal{X}$ is irreducible.

Proof. Throughout this proof, for a path η in a graph, let $s(\eta)$ and $t(\eta)$ be the starting state and the terminal state, respectively, of η in the graph. Also, for a state $v = v_1v_2\ldots v_\ell$ in $\mathcal{G}_{\mathcal{X}}$, $v \in \mathcal{V}^{(j)}$ is denoted by $v^{(j)}$ for $0 \leq j \leq T-1$.

Let $\mathcal{G}'$ be the graph defined in Step 4 of the MS algorithm. Consider the subgraph $\mathcal{H}$ of $\mathcal{G}'$ that is induced by the states in $\Sigma^\ell \setminus \mathcal{F}'$. Since $\Sigma^\ell \setminus \mathcal{F}' = \mathcal{B}_\ell(\mathcal{Y})$, all states in $\mathcal{H}$ have incoming edges and outgoing edges. Hence, $\mathcal{H}$ is a subgraph of $\mathcal{G}_{\mathcal{X}}$.

The key points of the proof are the following.

Claim 1: $\mathcal{H}$ is a presentation of $\mathcal{Y}$.

Claim 2: $\mathcal{H}$ is irreducible as a graph if[1] there exists a periodic bi-infinite sequence $\mathbf{y}$ in $\mathcal{Y}$ with a period p satisfying $p \equiv 1 \pmod{T}$.

Once these claims are proved, it is straightforward to check that the MS presentation $\mathcal{G}_{\mathcal{X}}$ of $\mathcal{X}$ is irreducible. Note that the graph $\mathcal{G}'$ is obtained from $\mathcal{H}$ by adding words in $\mathcal{F}^{(0)} = \mathcal{F}'$ to $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \ldots, \mathcal{V}^{(T-1)}$ and corresponding incoming and outgoing edges. Observe that (by Step 5 of the MS algorithm) a word $f' \in \mathcal{F}'$ is a state in $\mathcal{G}_{\mathcal{X}}$ if and only if there exist paths ρ_1, ρ_2 in $\mathcal{G}'$ satisfying $s(\rho_1) = f'$, $t(\rho_1) \in \Sigma^\ell \setminus \mathcal{F}'$ and $s(\rho_2) \in \Sigma^\ell \setminus \mathcal{F}'$, $t(\rho_2) = f'$. Since $\mathcal{H}$ is irreducible, $\mathcal{G}_{\mathcal{X}}$ is irreducible as well.

Proof of Claim 1. We need to show that $S(\mathcal{H}) \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq S(\mathcal{H})$. It is clear that $S(\mathcal{H}) \subseteq \mathcal{Y}$ since, by Remark 2.1, there is no path in $\mathcal{H}$ which generates words in $\mathcal{F}'$. Conversely, take an arbitrary bi-infinite sequence $\mathbf{x} = \ldots x_{-1}x_0x_1 \ldots \in \mathcal{Y}$. Since $f' \not\prec \mathbf{x}$ for every forbidden word $f' \in \mathcal{F}'$, we see that for any integer i, the states corresponding to $x_{i-\ell+1}x_{i-\ell+2}\ldots x_i$ are in $\mathcal{H}$. Therefore, there exists an edge labeled x_{i+1} from $x_{i-\ell+1}x_{i-\ell+2}\ldots x_i \in \mathcal{V}^{(j)}$ to $x_{i-\ell+2}\ldots x_ix_{i+1} \in \mathcal{V}^{(j+1 \mod T)}$ for all integers i and $0 \le j \le T-1$. Hence, $\mathbf{x} \in S(\mathcal{H})$, that is, $\mathcal{Y} \subseteq S(\mathcal{H})$.

Proof of Claim 2. A periodic bi-infinite sequence $\mathbf{y} \in \mathcal{Y}$ with period $p \equiv 1 \pmod{T}$ can be written as $\mathbf{y} = (y_1y_2\ldots y_n)^\infty$, for some $y_1y_2\ldots y_n \in \Sigma^n$, where n is some multiple of p satisfying $n \equiv 1 \pmod{T}$ and $n \ge \ell$.

As $\mathbf{y} \in \mathcal{Y}$, $y_{n-\ell+1}\ldots y_ny_1y_2\ldots y_n \in \mathcal{B}(\mathcal{Y})$. Thus, for every $i \in \{0, 1, \ldots, T-1\}$, there exists a path α in $\mathcal{H}$ satisfying $s(\alpha) = z^{(i)} = y_{n-\ell+1}\ldots y_n$ and generating $y_1y_2\ldots y_n$. Observe that $t(\alpha)$ is also $z^{(i')} = y_{n-\ell+1}\ldots y_n$ for some $i' \in \{0, 1, \ldots, T-1\}$. However, since $|y_1y_2\ldots y_n| = n \equiv 1 \pmod{T}$, we have $i' = i+1 \mod T$. This automatically implies that for the word $z = y_{n-\ell+1}\ldots y_n$ in $\mathcal{B}(\mathcal{Y})$, there is a path β_{jk} in $\mathcal{H}$ such that $s(\beta_{jk}) = z^{(j)}$ and $t(\beta_{jk}) = z^{(k)}$ for any ordered pair (j,k), where $0 \le j,k \le T-1$.

Now take an arbitrary pair of states $u^{(r)}$ and $v^{(s)}$ in $\mathcal{H}$. Since $\mathcal{Y}$ is irreducible, there exist words w' and w^* in $\mathcal{B}(\mathcal{Y})$ so that $uw'z$ and zw^*v are in $\mathcal{B}(\mathcal{Y})$. Thus, there exists a path γ generating $w'z$ such that $s(\gamma) = u^{(r)}$ and $t(\gamma) = z^{(j)}$ for some $0 \le j \le T-1$, and a path δ generating w^*v such that $s(\delta) = z^{(k)}$ for some $0 \le k \le T-1$ and $t(\delta) = v^{(s)}$. As there is a path β_{jk} from $z^{(j)}$ to $z^{(k)}$ from the argument above, we have a path $\gamma\beta_{jk}\delta$ starting from $u^{(r)}$ and terminating at $v^{(s)}$. Hence, the presentation $\mathcal{H}$ is irreducible as a graph. □

4. Periods in PFT's. Given a PFT $\mathcal{X}$, define its *descriptive period*, $T^{(\mathcal{X})}_{\text{desc}}$, to be the smallest integer among all T^* such that $\mathcal{X} = \mathcal{X}_{\{\mathcal{F}^*, T^*\}}$ for some $\mathcal{F}^*$. Note that $T^{(\mathcal{X})}_{\text{desc}} = 1$ iff $\mathcal{X}$ is in fact an SFT. Thus, if $\mathcal{X}$ is a proper PFT, then $T^{(\mathcal{X})}_{\text{desc}} \ge 2$.

The descriptive period is not the only notion of "period" that can be associated with a PFT. A bi-infinite sequence $\mathbf{x} = \ldots x_{-1}x_0x_1\ldots$ is said to be *periodic* if there exists a positive integer n such that $x_i = x_{i+n}$ for all $i \in \mathbb{Z}$. Any such integer n is called a *period* of the sequence $\mathbf{x}$, in which case we say that $\mathbf{x}$ has period n. Note that if $\mathbf{x}$ has period n, then it also has period $2n$, $3n$, etc. We define the *sequential period* of a PFT (or more generally, a sofic shift) $\mathcal{X}$ to be the smallest period of any periodic bi-infinite sequence in $\mathcal{X}$; we denote this by $T^{(\mathcal{X})}_{\text{seq}}$.

In the case when $\mathcal{X}$ is an *irreducible* PFT (or more generally, an irreducible sofic shift), we further define a "graphical period" as follows. Let $\mathcal{G}$ be a presentation of $\mathcal{X}$ with state set $\mathcal{V}(\mathcal{G}) = \{V_1, \ldots, V_r\}$. For each $V_i \in \mathcal{V}(\mathcal{G})$, define $\text{per}(V_i)$ to be the

[1] In fact, the converse is true as well.

greatest common divisor (gcd) of the lengths of paths (cycles) in $\mathcal{G}$ that begin and end at V_i, and further define $\text{per}(\mathcal{G}) = \gcd(\text{per}(V_1), \ldots, \text{per}(V_r))$. It is well known that when $\mathcal{G}$ is irreducible, $\text{per}(V_i) = \text{per}(V_j)$ for each pair of states $V_i, V_j \in \mathcal{V}(\mathcal{G})$, and hence $\text{per}(\mathcal{G}) = \text{per}(V)$ for any $V \in \mathcal{V}(\mathcal{G})$. The *graphical period*, $T_{\text{graph}}^{(\mathcal{X})}$, of $\mathcal{X}$ is defined to be the least $\text{per}(\mathcal{G})$ of any irreducible presentation $\mathcal{G}$ of $\mathcal{X}$.

In this section, we determine what relationships, if any, exist, between the descriptive, sequential and graphical periods of a PFT.

4.1. Comparing descriptive and sequential periods. The following proposition, proved in [6], will be useful for our initial development.

PROPOSITION 4.1 ([6], Proposition 1). *Let $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ be an irreducible, proper PFT, and let $\mathcal{G}$ be an irreducible presentation of $\mathcal{X}$. Then,* $\gcd(\mathit{per}(\mathcal{G}), T) \neq 1$.

We will later prove a sharper result (Theorem 4.14) for the case when $T = T_{\text{desc}}^{(\mathcal{X})}$. In any case, the proposition above allows us to prove the following result, which shows that a proper PFT $\mathcal{X}$ can have $T_{\text{desc}}^{(\mathcal{X})}$ arbitrarily larger than $T_{\text{seq}}^{(\mathcal{X})}$.

PROPOSITION 4.2. *Let $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ be a proper PFT with*

$$\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)}) = (\mathcal{F}', \emptyset, \ldots, \emptyset)$$

for some $\mathcal{F}' \subseteq \Sigma^\ell$, $\ell \geq 1$, and some prime T. Suppose that the SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ has the following properties:

(i) $\mathcal{Y}$ is irreducible;
(ii) $\mathcal{B}_\ell(\mathcal{Y}) = \Sigma^\ell \setminus \mathcal{F}'$; and
(iii) $a^\infty \in \mathcal{Y}$ for some $a \in \Sigma$.

Then, $T_{desc}^{(\mathcal{X})} = T$. (Since $\mathcal{Y} \subseteq \mathcal{X}$, (iii) above also implies that $T_{seq}^{(\mathcal{X})} = 1$.)

Proof. First observe that, by Theorem 3.1, the MS presentation, $\mathcal{G}_\mathcal{X}$, of $\mathcal{X}$ is irreducible, since the bi-infinite sequence $a^\infty \in \mathcal{Y}$ has period 1. Also, note that, by construction, the length of any cycle in $\mathcal{G}_\mathcal{X}$ is some multiple of T, and hence, $\text{per}(\mathcal{G}_\mathcal{X})$ must be kT for some $k \geq 1$. On the other hand, the cycle in $\mathcal{G}_\mathcal{X}$ that generates a^∞ is of length T, and passes through the states a^ℓ. Hence, by the irreducibility of $\mathcal{G}_\mathcal{X}$, we have that $\text{per}(\mathcal{G}_\mathcal{X}) = \text{per}(a^\ell) = T$.

Since $\mathcal{X}$ is proper, we have from Proposition 4.1 that $\gcd(\text{per}(\mathcal{G}_\mathcal{X}), T^*) \neq 1$ for all T^* satisfying $\mathcal{X} = \mathcal{X}_{\{\mathcal{F}^*,T^*\}}$. As T is prime, $\gcd(\text{per}(\mathcal{G}_\mathcal{X}), T') = \gcd(T, T') = 1$ for all $T' < T$. Therefore, T is the descriptive period of $\mathcal{X}$. □

As a concrete application of the proposition above, we prove the following result.

COROLLARY 4.3. *Let $\Sigma = \{0, 1\}$. For the PFT $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$, with T an arbitrary prime, and $\mathcal{F} = (\{11\}, \emptyset, \ldots, \emptyset)$, we have $T_{seq}^{(\mathcal{X})} = 1$ and $T_{desc}^{(\mathcal{X})} = T$.*

Proof. Let $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ be the SFT over Σ with forbidden set $\mathcal{F}' = \{11\}$. It is easy to verify that $\mathcal{Y}$ is irreducible, $\mathcal{B}_2(\mathcal{Y}) = \Sigma^2 \setminus \mathcal{F}'$, and $0^\infty \in \mathcal{Y}$. Since $\mathcal{Y} \subseteq \mathcal{X}$, we have $0^\infty \in \mathcal{X}$, and hence, $T_{\text{seq}}^{(\mathcal{X})} = 1$.

We claim that $\mathcal{X}$ is a proper PFT, and hence, by Proposition 4.2, $T_{\text{desc}}^{(\mathcal{X})} = T$. To see this, suppose to the contrary that $\mathcal{X} = \mathcal{Y}_{\mathcal{F}''}$ for some SFT $\mathcal{Y}_{\mathcal{F}''}$ with forbidden set $\mathcal{F}''$. We may assume that $\mathcal{F}'' \subseteq \Sigma^\ell$ for some $\ell \geq 1$. Note that the bi-infinite sequence $\mathbf{w} = 0^\infty 110^\infty$ is in $\mathcal{X}$. Therefore, none of the length-ℓ subwords of $\mathbf{w}$ is in $\mathcal{F}''$. Now, the sequence

$$\mathbf{w}' = 0^\infty (110^{T\ell-1})^T 0^\infty$$

has the same set of length-ℓ subwords as $\mathbf{w}$, and since none of these words is in $\mathcal{F}''$, we must have $\mathbf{w}' \in \mathcal{Y}_{\mathcal{F}''} = \mathcal{X}$. But this is impossible as the T instances of 11 in $\mathbf{w}'$ are separated such that, no matter how $\mathbf{w}'$ is shifted, there is always an instance of 11 beginning at some index $i \equiv 0 \pmod{T}$. This contradiction shows that $\mathcal{X}$ must be a proper PFT. □

Having shown that it is possible for $T_{\text{desc}}^{(\mathcal{X})}$ to be arbitrarily larger than $T_{\text{seq}}^{(\mathcal{X})}$ for a PFT $\mathcal{X}$, we present an example of the opposite phenomenon next.

Set $\Sigma = \{0, 1\}$ and let $\oplus$ denote modulo-2 addition. We define a sliding-block map ψ as follows: for a non-empty word $u = u_1u_2 \ldots u_n \in \Sigma^n$, (resp. a bi-infinite sequence $\mathbf{w} = \ldots w_{-1}w_0w_1 \ldots$ over Σ), define $\psi(u) = u_1^*u_2^* \ldots u_{n-1}^*$, where $u_i^* = u_i \oplus u_{i+1}$ for $1 \le i \le n-1$ (resp. $\psi(\mathbf{w}) = \ldots w_{-1}^*w_0^*w_1^* \ldots$, where $w_i^* = w_i \oplus w_{i+1}$ for each i). By convention, $\psi(u) = \epsilon$ when $u \in \Sigma^1$. For $k \ge 1$, consider the PFT $\mathcal{X}_k = \mathcal{X}_{\{\mathcal{F}_k, 2\}}$, with $\mathcal{F}_k = (\mathcal{F}_k^{(0)}, \mathcal{F}_k^{(1)})$ defined as follows:

- $\mathcal{F}_k^{(1)} = \emptyset$ for all $k \ge 1$.
- $\mathcal{F}_1^{(0)} = \{0\}$, and for $k \ge 2$, we set $\mathcal{F}_k^{(0)} = \psi^{-1}(\mathcal{F}_{k-1}^{(0)})$. That is, $\mathcal{F}_k^{(0)}$ is the inverse image of $\mathcal{F}_{k-1}^{(0)}$ under ψ.

It is easy to see that for each $k \ge 1$, every word $f \in \mathcal{F}_k^{(0)}$ has length $|f| = k$, and in particular, we have $0^k \in \mathcal{F}_k^{(0)}$. Moreover, as ψ is a two-to-one mapping, we have $|\mathcal{F}_k^{(0)}| = 2^{k-1}$.

In the remainder of this subsection, we will show that the $\mathcal{X}_k$'s form a class of PFT's whose sequential period can be arbitrarily large compared to its descriptive period (see Theorem 4.12 below). We begin with a useful observation concerning the map ψ.

Proposition 4.4. *For a binary word $u = u_1u_2 \ldots u_r$ of length $r > m$, let $u_1^*u_2^* \ldots u_{r-m}^* = \psi^m(u)$. If $m = 2^j$ for some $j \ge 0$, then $u_i^* = u_i \oplus u_{i+2^j}$ for $1 \le i \le r-m$. Furthermore, if $m = 2^j - 1$ for some $j \ge 0$, then $u_i^* = u_i \oplus u_{i+1} \oplus \cdots \oplus u_{i+2^j-1}$ for $1 \le i \le r-m$.*

Proof. The case of $j = 0$ is trivial for both statements, so suppose that these statements hold when $j = n \ge 0$. Then when $j = n+1$, $\psi^{2^{n+1}}(u) = \psi^{2^n}(\psi^{2^n}(u)) = u_1^*u_2^* \ldots u_{r-2^{n+1}}^*$, where u_i^* is given by $(u_i \oplus u_{i+2^n}) \oplus (u_{i+2^n} \oplus u_{i+2^n+2^n}) = u_i \oplus u_{i+2^{n+1}}$ for $1 \le i \le r - 2^{n+1}$. Similarly, $\psi^{2^{n+1}-1}(u) = \psi^{2^n}(\psi^{2^n-1}(u)) = u_1^*u_2^* \ldots u_{r-2^{n+1}+1}^*$, where $u_i^* = (u_i \oplus u_{i+1} \oplus \cdots \oplus u_{i+2^n-1}) \oplus (u_{i+2^n} \oplus u_{i+2^n+1} \oplus \cdots \oplus u_{i+2^n+2^n-1}) = u_i \oplus u_{i+1} \oplus \cdots \oplus u_{i+2^{n+1}-1}$ for $1 \le i \le r - 2^{n+1} + 1$. The proposition follows by induction. □

The corollary below simply follows from the fact that for any $f \in \mathcal{F}_k^{(0)}$, we must have $\psi^{k-1}(f) = 0$.

Corollary 4.5. *If $z \in \Sigma^{2^j}$, $j \ge 0$, has an odd number of 1's, then $z \notin \mathcal{F}_{2^j}^{(0)}$.*

We next prove some important facts about the PFT's $\mathcal{X}_k$.

Proposition 4.6. *For $k \ge 1$, (a) $\mathcal{X}_{k+1} = \psi^{-1}(\mathcal{X}_k)$, and (b) $\mathcal{X}_k$ is a proper PFT.*

Proof. (a) Let $\mathbf{x}$ and $\mathbf{x}'$ be bi-infinite sequences such that $\mathbf{x}' \in \psi^{-1}(\mathbf{x})$, *i.e.*, $\psi(\mathbf{x}') = \mathbf{x}$. Observe that, for any word $f \in \Sigma^{\ell}$ ($\ell \geq 0$) and any integer i, we have $f \prec_i \mathbf{x}$ iff there exists $f' \in \Sigma^{\ell+1}$ such that $\psi(f') = f$ and $f' \prec_i \mathbf{x}'$. In particular, $f \prec_i \mathbf{x}$ for some $f \in \mathcal{F}_k^{(0)}$ iff $f' \prec_i \mathbf{x}'$ for some $f' \in \psi^{-1}(\mathcal{F}_k^{(0)}) = \mathcal{F}_{k+1}^{(0)}$. It follows that $\mathbf{x} \in \mathcal{X}_k$ iff $\mathbf{x}' \in \mathcal{X}_{k+1}$, from which we obtain (a).

(b) Suppose to the contrary that $\mathcal{X}_k$ is not a proper PFT for some $k \geq 1$. Then, $\mathcal{X}_k = \mathcal{Y}$ for some SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$, where every forbidden word in $\mathcal{F}'$ has the same length, ℓ. Pick a $j \geq 0$ such that $2^j \geq k$, and set $r = 2^j - k$. By (a) above, $\mathcal{X}_{2^j} = \psi^{-r}(\mathcal{X}_k) = \psi^{-r}(\mathcal{Y})$. Note that $\psi^{-r}(\mathcal{Y})$ is also an SFT, with forbidden set $\psi^{-r}(\mathcal{F}')$. All words in $\psi^{-r}(\mathcal{F}')$ have length $\ell' = \ell + r$.

For the PFT $\mathcal{X}_{2^j}$, observe that the bi-infinite sequence

$$\mathbf{w} = (0^{2^j-1}1)^{\infty}0^{2^j}(10^{2^j-1})^{\infty}$$

is in $\mathcal{X}_{2^j}$ as $\mathbf{w}$ contains a word in $\mathcal{F}_{2^j}^{(0)}$ (*i.e.*, 0^{2^j}) only once, by Corollary 4.5. Therefore, every subword of $\mathbf{w}$ is in $\mathcal{B}(\mathcal{X}_{2^j}) = \mathcal{B}(\psi^{-r}(\mathcal{Y}))$.

Now, consider the bi-infinite sequence

$$\mathbf{w}' = (0^{2^j-1}1)^{\infty}0^{2^j}(10^{2^j-1})^{2\ell'+1}10^{2^j}(10^{2^j-1})^{\infty}.$$

Note that every length-ℓ' subword of $\mathbf{w}'$ is also a subword of $\mathbf{w}$, and hence, is in $\mathcal{B}(\psi^{-r}(\mathcal{Y}))$. This implies that $\mathbf{w}' \in \psi^{-r}(\mathcal{Y})$. For the two distinct indices m, n ($m < n$) such that $0^{2^j} \prec_m \mathbf{w}'$ and $0^{2^j} \prec_n \mathbf{w}'$, we have $n - m = 2^j(2\ell' + 2) + 1$, so that $m \not\equiv n \pmod 2$. But, since $0^{2^j} \in \mathcal{F}_{2^j}^{(0)}$, this implies that $\mathbf{w}' \notin \mathcal{X}_{2^j}$, which is a contradiction. □

Statement (b) of Proposition 4.6 implies that $T_{\mathrm{desc}}^{(\mathcal{X}_k)} = 2$ for all $k \geq 1$. In contrast, the following theorem shows that $T_{\mathrm{seq}}^{(\mathcal{X}_k)} \to \infty$ as $k \to \infty$.

THEOREM 4.7. *For any $j \geq 0$ and $2^j + 1 \leq k \leq 2^{j+1}$, the periods of periodic sequences in $\mathcal{X}_k$ must be multiples of 2^{j+1}.*

To prove Theorem 4.7, we need the next three lemmas.

LEMMA 4.8. *If $\mathbf{x} \in \{0,1\}^{\mathbb{Z}}$ is a periodic sequence, then so is $\psi(\mathbf{x})$. Furthermore, any period of $\mathbf{x}$ is also a period of $\psi(\mathbf{x})$.*

Proof. If $\mathbf{x}$ is periodic with a period n, then $\mathbf{x}$ can be written as $\mathbf{x} = (x_1x_2\ldots x_n)^{\infty}$ for some $x_1x_2\ldots x_n \in \Sigma^n$. Then, it is clear that $\psi(\mathbf{x}) = (z_1z_2\ldots z_n)^{\infty}$, where $z_i = x_i \oplus x_{i+1}$ for $1 \leq i \leq n-1$ and $z_n = x_n \oplus x_1$. Hence, $\psi(\mathbf{x})$ is periodic and has a period n as well. □

LEMMA 4.9. *Let $j \geq 0$. For the PFT $\mathcal{X}_{2^j+1}$,*

$$\mathcal{F}_{2^j+1}^{(0)} = \{f^*f_1^* : f^* = f_1^*f_2^*\ldots f_{2^j}^* \in \Sigma^{2^j}\}.$$

Proof. Recall that for a word $f = f_1f_2\ldots f_{2^j+1} \in \mathcal{F}_{2^j+1}^{(0)}$, we have $\psi^{2^j}(f) = 0$. Since Proposition 4.4 shows that $\psi^{2^j}(f) = f_1 \oplus f_{2^j+1}$, we have $f_1 = f_{2^j+1}$. Since $|\mathcal{F}_{2^j+1}^{(0)}| = 2^{2^j} = |\Sigma^{2^j}|$, we have $\mathcal{F}_{2^j+1}^{(0)} = \{f^*f_1^* : f^* = f_1^*f_2^*\ldots f_{2^j}^* \in \Sigma^{2^j}\}$ as required. □

LEMMA 4.10. *For $j \geq 0$, there is no periodic sequence $\mathbf{x}$ in $\mathcal{X}_{2^j+1}$ whose period is $(2t+1)2^j$ for some $t \geq 0$.*

Proof. When $j = 0$, $\mathcal{X}_{2^j+1} = \mathcal{X}_2$, and it may be verified that there is no periodic sequence in $\mathcal{X}_2$ with an odd period. So let $j \geq 1$, and assume to the contrary that there exists a periodic sequence $\mathbf{x} = \ldots x_{-1}x_0x_1 \ldots \in \mathcal{X}_{2^j+1}$ whose period is $(2t+1)2^j$ for some $t \geq 0$. Then, $\mathbf{x}$ is of the form $(x_0x_1 \ldots x_{(2t+1)2^j-1})^\infty$. Without loss of generality, we may assume that for every even integer i, $u \prec_i \mathbf{x}$ implies $u \notin \mathcal{F}^{(0)}_{2^j+1}$. Then, for each integer m, $x_{m2^j}x_{m2^j+1} \ldots x_{(m+1)2^j} \notin \mathcal{F}^{(0)}_{2^j+1}$. Since $\mathcal{F}^{(0)}_{2^j+1} = \{f^* f_1^* \;:\; f^* \in \Sigma^{2^j}\}$ from Lemma 4.9, we have $x_{m2^j} \neq x_{(m+1)2^j}$. This implies $x_0 = x_{(2t)2^j}$ as $|\Sigma| = 2$. But then, $x_{(2t)2^j} \ldots x_{(2t+1)2^j-1}x_0 \in \mathcal{F}^{(0)}_{2^j+1}$, which is a contradiction. ☐

We are now in a position to prove Theorem 4.7.

Proof of Theorem 4.7. To prove the theorem, it is enough to show that for $j \geq 0$, the periods of periodic sequences in $\mathcal{X}_{2^j+1}$ must be multiples of 2^{j+1}. It then follows, by Lemma 4.8, that the same also applies to periodic sequences in $\mathcal{X}_k$, for $2^j + 1 < k \leq 2^{j+1}$.

When $j = 0$, the required statement clearly holds by Lemma 4.10. So, suppose that the statement is true for some $j \geq 0$, so that periodic sequences in $\mathcal{X}_{2^{j+1}}$ have only multiples of 2^{j+1} as periods. Therefore, by Lemma 4.8, periodic sequences in $\mathcal{X}_{2^{j+1}+1}$ also can only have multiples of 2^{j+1} as periods. However, by Lemma 4.10, no periodic sequence in $\mathcal{X}_{2^{j+1}+1}$ can have an odd multiple of 2^{j+1} as a period. Hence, all periodic sequences in $\mathcal{X}_{2^{j+1}+1}$ have periods that are multiples of 2^{j+2}. The theorem follows by induction. ☐

Theorem 4.7 shows that for $2^j + 1 \leq k \leq 2^{j+1}$, we have $T^{(\mathcal{X}_k)}_{\text{seq}} \geq 2^{j+1}$. To put this another way, we have $T^{(\mathcal{X}_k)}_{\text{seq}} \geq 2^{\lceil \log_2 k \rceil}$. In fact, this holds with equality.

Corollary 4.11. *$T^{(\mathcal{X}_1)}_{seq} = 1$, and for $k \geq 2$, if $j \geq 0$ is such that $2^j + 1 \leq k \leq 2^{j+1}$, then $T^{(\mathcal{X}_k)}_{seq} = 2^{j+1}$.*

Proof. When $k = 1$, $T^{(\mathcal{X}_1)}_{\text{seq}} = 1$ as $1^\infty \in \mathcal{X}_1$. So let $k \geq 2$, and let $j \geq 0$ be such that $2^j + 1 \leq k \leq 2^{j+1}$. We only need to show that $T^{(\mathcal{X}_k)}_{\text{seq}} \leq 2^{j+1}$. The bi-infinite sequence $\mathbf{w} = (0^{2^{j+1}-1}1)^\infty$ is in $\mathcal{X}_{2^{j+1}}$ since, by Corollary 4.5, $\mathbf{w}$ contains no word in $\mathcal{F}^{(0)}_{2^{j+1}}$ as a subword. Since $\mathbf{w}$ has period 2^{j+1}, by Lemma 4.8, $\mathbf{w}' = \psi^{2^{j+1}-k}(\mathbf{w}) \in \mathcal{X}_k$ has period 2^{j+1} as well. Thus, $T^{(\mathcal{X}_k)}_{\text{seq}} \leq 2^{j+1}$. ☐

We have thus proved the following result.

Theorem 4.12. *For each $k \geq 1$, $T^{(\mathcal{X}_k)}_{desc} = 2$, while $T^{(\mathcal{X}_k)}_{seq} = 2^{\lceil \log_2 k \rceil}$.*

From Proposition 4.2 and Theorem 4.12, we see that the descriptive period and the sequential period of a PFT can each be arbitrarily larger than the other.

4.2. Comparing descriptive and graphical periods. The PFT's $\mathcal{X}_k$ defined in the previous subsection also illustrate the fact that the graphical period of a PFT can be larger than its descriptive period. Of course, in order to make such a claim, we have to be able to show that the $\mathcal{X}_k$'s are irreducible so that their graphical periods can be defined. In fact, it turns out that $\mathcal{X}_k$ is irreducible iff $1 \leq k \leq 6$. To preserve the flow of our presentation, we prove this fact in Appendix A.

Thus, the graphical period of $\mathcal{X}_k$ is defined only for $1 \leq k \leq 6$. For these values of k, we have the following result.

Proposition 4.13. *$T^{(\mathcal{X}_k)}_{graph} \geq 2^{\lceil \log_2 k \rceil}$ holds when $1 \leq k \leq 6$.*

Proof. We must show that $T^{(\mathcal{X}_k)}_{\text{graph}} \geq T^{(\mathcal{X}_k)}_{\text{seq}}$. Since $\mathcal{X}_1$ is proper, $T^{(\mathcal{X}_1)}_{\text{graph}} \geq 2$ by Proposition 4.1. Thus, $T^{(\mathcal{X}_1)}_{\text{graph}} > T^{(\mathcal{X}_1)}_{\text{seq}} = 1$. So, let $k \geq 2$ and suppose $2^j + 1 \leq k \leq 2^{j+1}$ for some $j \geq 0$. By Corollary 4.11, we have $T^{(\mathcal{X}_k)}_{\text{seq}} = 2^{j+1}$. On the other hand, for any irreducible presentation $\mathcal{G}$ of $\mathcal{X}_k$, we have $\text{per}(\mathcal{G}) \geq 2^{j+1}$. Indeed, for each vertex V in $\mathcal{G}$, we have $\text{per}(V)$ being a multiple of 2^{j+1}; otherwise, we would have a contradiction of Theorem 4.7. Hence, $T^{(\mathcal{X}_k)}_{\text{graph}} \geq 2^{j+1} = T^{(\mathcal{X}_k)}_{\text{seq}}$ as required. □

Proposition 4.13 shows that $T^{(\mathcal{X}_k)}_{\text{graph}}$ is strictly larger than $T^{(\mathcal{X}_k)}_{\text{desc}} = 2$ when $3 \leq k \leq 6$. Equality can hold in Proposition 4.13 — for example, when $k = 2$. Since $\mathcal{X}_2$ is proper, and the MS presentation, $\mathcal{G}_{\mathcal{X}_2}$, of $\mathcal{X}_2$ is irreducible and $\text{per}(\mathcal{G}_{\mathcal{X}_2}) = 2$, we have $T^{(\mathcal{X}_2)}_{\text{graph}} = 2$. Furthermore, $T^{(\mathcal{X}_2)}_{\text{seq}} = 2$ from Corollary 4.11. Thus, $\mathcal{X}_2$ is an example of a proper PFT $\mathcal{X}$ in which $T^{(\mathcal{X})}_{\text{seq}} = T^{(\mathcal{X})}_{\text{graph}} = T^{(\mathcal{X})}_{\text{desc}}$ holds.

We have thus found that, for an irreducible PFT $\mathcal{X}$, $T^{(\mathcal{X})}_{\text{graph}}$ can be larger than $T^{(\mathcal{X})}_{\text{desc}}$. However, the converse does not hold; in fact, $T^{(\mathcal{X})}_{\text{desc}}$ always divides $T^{(\mathcal{X})}_{\text{graph}}$.

THEOREM 4.14. *Let $\mathcal{X}$ be an irreducible PFT, and $\mathcal{G}$ be any irreducible presentation of $\mathcal{X}$. Then,* $\gcd(\mathit{per}(\mathcal{G}), T_{desc}) = T_{desc}$ *holds. Thus, for any irreducible PFT $\mathcal{X}$, $T^{(\mathcal{X})}_{desc}$ always divides $T^{(\mathcal{X})}_{graph}$.*

Note that Theorem 4.14 sharpens Proposition 4.1 in the case when $T = T^{(\mathcal{X})}_{\text{desc}}$. To prove the theorem, we need the following key lemma, the proof of which we defer to the end of this section.

LEMMA 4.15. *Let $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ be an irreducible PFT with period T, and $\mathcal{G}$ be an irreducible presentation of $\mathcal{X}$. If* $\gcd(\mathit{per}(\mathcal{G}), T) = d$, *then for any state V in $\mathcal{G}$, there exists a cycle C with length L, starting and terminating at V, such that* $\gcd(L, T) = d$.

Using Lemma 4.15, we can prove Theorem 4.14 as follows.

Proof of Theorem 4.14. For a bi-infinite sequence $\mathbf{x} = \ldots x_{-1}x_0x_1\ldots$, let us denote the left-infinite subsequence $\ldots x_{m-1}x_m$ and the right-infinite subsequence $x_m x_{m+1}\ldots$ of $\mathbf{x}$ by $\mathbf{x}_{m^-}$ and $\mathbf{x}_{m^+}$, respectively.

Let $T^{(\mathcal{X})}_{\text{desc}} = T$, and represent $\mathcal{X}$ as $\mathcal{X}_{\{\mathcal{F},T\}}$, where $\mathcal{F}$ is in standard form, *i.e.*, $\mathcal{F} = (\mathcal{F}^{(0)}, \emptyset, \ldots, \emptyset)$ and each $f \in \mathcal{F}^{(0)}$ has length ℓ. Contrary to the statement in the theorem, assume that $\gcd(\text{per}(\mathcal{G}), T) = d < T$. In this case, we consider the PFT $\widetilde{\mathcal{X}} = \mathcal{X}_{\{\widetilde{\mathcal{F}},T\}}$, with $\widetilde{\mathcal{F}} = (\tilde{\mathcal{F}}^{(0)}, \tilde{\mathcal{F}}^{(1)}, \ldots, \tilde{\mathcal{F}}^{(T-1)})$ defined as follows:

$$\tilde{\mathcal{F}}^{(k)} = \begin{cases} \mathcal{F}^{(0)} & \text{if } \ k \equiv 0 \pmod{d}. \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to see that $\widetilde{\mathcal{X}} = \mathcal{X}_{\{\widehat{\mathcal{F}},d\}}$, where $\widehat{\mathcal{F}} = (\tilde{\mathcal{F}}^{(0)}, \tilde{\mathcal{F}}^{(1)}, \ldots, \tilde{\mathcal{F}}^{(d-1)})$. Thus, the theorem would be proved if we can show that $\widetilde{\mathcal{X}} = \mathcal{X}$, as we would then have $\mathcal{X} = \mathcal{X}_{\{\widehat{\mathcal{F}},d\}}$, and hence, $T^{(\mathcal{X})}_{\text{desc}} \leq d < T$, which contradicts our assumption on T.

The remainder of this proof is devoted to showing that $\widetilde{\mathcal{X}} = \mathcal{X}$. It is clear that $\widetilde{\mathcal{X}} \subseteq \mathcal{X}$. Suppose that $\widetilde{\mathcal{X}} \neq \mathcal{X}$. We will show that this leads to a contradiction.

Since $\widetilde{\mathcal{X}} \subsetneq \mathcal{X}$, there exists a bi-infinite sequence $\mathbf{x} \in \mathcal{X} \setminus \widetilde{\mathcal{X}}$. Since $\mathbf{x} \notin \widetilde{\mathcal{X}}$, for each $r = 0, 1, \ldots, d-1$, we have $f_r \prec_j \sigma^r(\mathbf{x})$ for some $f_r \in \mathcal{F}^{(0)}$ and $j \equiv 0 \pmod{d}$. That is, for each $r \in \{0, 1, \ldots, d-1\}$, there exists $f_r \in \mathcal{F}^{(0)}$ such that $f_r \prec_{j_r} \mathbf{x}$ for some $j_r \equiv r \pmod{d}$.

Consider f_0, and let j_0 be an index such that $f_0 \prec_{j_0} \mathbf{x}$ and $j_0 \equiv 0 \pmod{d}$. As $\mathbf{x} \in \mathcal{X}$, there is a path α in $\mathcal{G}$ generating that f_0 as a subword of $\mathbf{x}$. Let U and V denote the initial state and the terminal state of α, respectively. Since $\mathcal{G}$ is irreducible, there is a path β with the initial state V and the terminal state U. From the fact that $\gcd(\mathrm{per}(\mathcal{G}), T) = d$, we have that $|\alpha| + |\beta|$ is a multiple of d since the path $\alpha\beta$ is a cycle in $\mathcal{G}$. Also, from Lemma 4.15, there is a cycle C with length L, starting and terminating at U, such that $\gcd(L, T) = d$.

Let u_0 and v_0 be the words generated by β and C, respectively. Then, by using paths α, β and C appropriately, we can generate the bi-infinite sequence $\mathbf{x}^{(0)}$ from $\mathcal{G}$ such that

$$\mathbf{x}^{(0)} = \mathbf{x}_{(j_0-1)^-}(f_0 u_0 (v_0)^{N_0})^{T/d} f_0 \mathbf{x}_{(j_0+\ell)^+},$$

where N_0 is a positive integer satisfying $|f_0| + |u_0| + N_0|v_0| \equiv d \pmod{T}$. Such an integer N_0 can always be found. Indeed, since $\gcd(|v_0|, T) = \gcd(L, T) = d$, there exists a positive integer m_0 such that $m_0|v_0| \equiv d \pmod{T}$. Moreover, $|f_0| + |u_0| = |\alpha| + |\beta|$ is a multiple of d; say, $|f_0| + |u_0| = m_1 d$. Take $N_0 = m_0 m_2$ for any positive integer $m_2 \equiv (1 - m_1) \pmod{T}$.

Observe that, by construction, the T/d instances of f_0 within $(f_0 u_0 (v_0)^{N_0})^{T/d}$ occur as subwords of $\mathbf{x}^{(0)}$ starting at certain indices $i_0^{(0)}, i_1^{(0)}, \ldots, i_{T/d-1}^{(0)}$ satisfying $i_s^{(0)} \equiv j_0 + sd \pmod{T}$, $0 \le s \le T/d - 1$. Furthermore, since $\mathbf{x}$ has $f_r \in \mathcal{F}^{(0)}$, $1 \le r \le d-1$, at an index $j_r \equiv r \pmod{d}$, we have $f_r \prec_{j_r'} \mathbf{x}^{(0)}$ for some $j_r' \equiv j_r \pmod{T}$. More precisely, $j_r' = j_r$ if $j_r < j_0$ and $j_r' = j_r + pT$ if $j_r > j_0$, where p is the integer such that $pd = |f_0| + |u_0| + N_0|v_0|$. Therefore, using the same argument described above, we can generate the bi-infinite sequence

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)}_{(j_1'-1)^-}(f_1 u_1 (v_1)^{N_1})^{T/d} f_1 \mathbf{x}^{(0)}_{(j_1'+\ell)^+}$$

from $\mathcal{G}$ so that the T/d f_1's in $(f_1 u_1 (v_1)^{N_1})^{T/d}$ have indices $i_0^{(1)}, i_1^{(1)}, \ldots, i_{T/d-1}^{(1)}$ satisfying $i_s^{(1)} \equiv j_1' + sd \equiv j_1 + sd \pmod{T}$, $0 \le s \le T/d - 1$.

Continue this procedure d times, once for each f_r, $r = 0, 1, \ldots, d-1$, and consider the resulting bi-infinite sequence $\mathbf{x}^{(d-1)} = \tilde{\mathbf{x}}$ generated from $\mathcal{G}$. Observe that for each $0 \le r \le d-1$, $\tilde{\mathbf{x}}$ contains f_r's at indices $i_0^{(r)}, i_1^{(r)}, \ldots, i_{T/d-1}^{(r)}$ satisfying $i_s^{(r)} \equiv j_r + sd \pmod{T}$, $0 \le s \le T/d - 1$. Equivalently, for each $r' = 0, 1, \ldots, T-1$, $\tilde{\mathbf{x}}$ contains some $f \in \mathcal{F}^{(0)}$ at an index $i_{r'} \equiv r' \pmod{T}$. Then, for each $r' = 0, 1, \ldots, T-1$, $\sigma^{r'}(\tilde{\mathbf{x}})$ contains some $f \in \mathcal{F}^{(0)}$ at an index $i \equiv 0 \pmod{T}$, and hence, $\tilde{\mathbf{x}} \notin \mathcal{X}$. However, $\tilde{\mathbf{x}}$ must be in $\mathcal{X}$ since $\tilde{\mathbf{x}}$ is generated from $\mathcal{G}$, which is the desired contradiction. Therefore, $\mathcal{X} = \widetilde{\mathcal{X}}$. □

To complete the proof of Theorem 4.14, it remains to prove Lemma 4.15. We need an elementary number-theoretic result to furnish a proof of the lemma. This is an extension of the well-known fact (see, for example, [3, p. 119]) that if a, b are two positive integers, with $b < a$ and $\gcd(a, b) = d$, then there exist non-negative integers $m, n \le a$ such that $ma - nb = d$.

PROPOSITION 4.16. *Let $a_1, a_2, \ldots, a_r$, $r \ge 2$, be positive integers, with $0 < a_r < \cdots < a_2 < a_1$ and $\gcd(a_1, a_2, \ldots, a_r) = 1$. Then, there exist non-negative integers $N_1, N_2, \ldots, N_r \le a_1^{r-1}$ such that $N_1 a_1 - N_2 a_2 + \cdots + (-1)^{r-1} N_r a_r = 1$.*

Proof. Recall that $\gcd(a_1, a_2, \ldots, a_r) = \gcd(a_1, \gcd(a_2, \ldots, a_r))$. Also observe that $a_1 > \gcd(a_2, \ldots, a_r)$ as $a_1 > a_r$. Therefore, there exist $0 \le m_1, n_1 \le a_1$ such that

$$m_1 a_1 - n_1 \times \gcd(a_2, \ldots, a_r) = 1. \tag{1}$$

Similarly, for $\gcd(a_2, \ldots, a_r)$, there exist $0 \le m_2, n_2 \le a_2$ such that

$$m_2 a_2 - n_2 \times \gcd(a_3, \ldots, a_r) = \gcd(a_2, \ldots, a_r). \tag{2}$$

Substituting (2) into (1), we have

$$m_1 a_1 - n_1 m_2 a_2 + n_1 n_2 \times \gcd(a_3, \ldots, a_r) = 1,$$

where $m_1, n_1 m_2, n_1 n_2 \le a_1^2$. The proposition follows by continuing this procedure. ☐

From Proposition 4.16, we derive the following corollary.

COROLLARY 4.17. *Let $a_1, a_2, \ldots, a_r$ be r positive integers, where $0 < a_r < \cdots < a_2 < a_1$, such that $\gcd(a_1, a_2, \ldots, a_r) = 1$. Then, for each integer $M \ge a_1^r(a_1 + a_2 + \cdots + a_r)$, there exist $h_1, h_2, \ldots, h_r \ge 0$ such that $M = h_1 a_1 + h_2 a_2 + \cdots + h_r a_r$.*

Proof. It is enough to show that the statement is true for $M_p = a_1^r(a_1 + a_2 + \cdots + a_r) + p$, $0 \le p < a_1$, since for each $M \ge a_1^r(a_1 + a_2 + \cdots + a_r) + a_1$, there exists M_p such that $M - M_p = t a_1$ for some $t \ge 1$.

Now, $M_p = a_1^r(a_1 + a_2 + \cdots + a_r) + p \times 1 = a_1^r(a_1 + a_2 + \cdots + a_r) + p(N_1 a_1 - N_2 a_2 + \cdots + (-1)^{r-1} N_r a_r)$, where $N_1, N_2, \ldots, N_r \le a_1^{r-1}$, by Proposition 4.16. Hence, $M_p = (a_1^r + pN_1)a_1 + (a_1^r - pN_2)a_2 + \cdots + (a_1^r + (-1)^{r-1} pN_r)a_r$ and the coefficient of a_i is non-negative for each i. ☐

We are now in a position to prove Lemma 4.15.

Proof of Lemma 4.15. Since the presentation $\mathcal{G}$ is irreducible, $\mathrm{per}(\mathcal{G}) = \mathrm{per}(V)$ for any vertex V in $\mathcal{G}$. Furthermore, $\mathrm{per}(V) = \gcd(c_1, c_2, \ldots, c_r)$, where each c_i represents the length of a simple cycle C_i starting and terminating at V. Without loss of generality, we can assume that

1. $c_i \ne c_j$ for $1 \le i, j \le r$, $i \ne j$;
2. $c_1 > c_2 > \cdots > c_r$.

Since $\gcd(\mathrm{per}(\mathcal{G}), T) = \gcd(\mathrm{per}(V), T) = d$, $\mathrm{per}(V) = Pd$ and $T = Qd$ for some $P, Q \ge 1$ such that $\gcd(P, Q) = 1$. Then, $\gcd(c_1/(Pd), c_2/(Pd), \ldots, c_r/(Pd)) = 1$. Set $c_i/(Pd) = a_i$ for $1 \le i \le r$. Pick an integer $M \ge a_1^r(a_1 + a_2 + \cdots + a_r)$ such that $\gcd(M, Q) = 1$. For such an integer M, we see, from Corollary 4.17, that there exist $h_1, h_2, \ldots, h_r \ge 0$ such that $M = h_1 a_1 + h_2 a_2 + \cdots + h_r a_r$. Let C be the cycle, starting and terminating at V, which is generated by passing through each simple cycle C_i h_i times. The length L of the cycle C is given by $L = h_1 c_1 + h_2 c_2 + \cdots + h_r c_r = Pd(h_1 a_1 + h_2 a_2 + \cdots + h_r a_r) = PMd$. Observe that $\gcd(L, T) = \gcd(PMd, Qd) = d \times \gcd(PM, Q) = d$, as $\gcd(PM, Q) = 1$. ☐

4.3. Comparing sequential and graphical periods. In this subsection, we present examples of irreducible PFT's $\mathcal{X}$ where we have $T_{\mathrm{seq}}^{(\mathcal{X})} \ll T_{\mathrm{graph}}^{(\mathcal{X})}$, and other examples where we have $T_{\mathrm{seq}}^{(\mathcal{X})} \gg T_{\mathrm{graph}}^{(\mathcal{X})}$.

Examples of the former phenomenon can be obtained from the following strengthening of Proposition 4.2.

PROPOSITION 4.18. *Let $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ be a proper PFT with*

$$\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)}) = (\mathcal{F}', \emptyset, \ldots, \emptyset)$$

for some $\mathcal{F}' \subseteq \Sigma^\ell$, $\ell \geq 1$, and some prime T. Suppose that the SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ has the following properties:

(i) $\mathcal{Y}$ is irreducible;
(ii) $\mathcal{B}_\ell(\mathcal{Y}) = \Sigma^\ell \setminus \mathcal{F}'$; and
(iii) $a^\infty \in \mathcal{Y}$ for some $a \in \Sigma$.

Then, $T^{(\mathcal{X})}_{desc} = T^{(\mathcal{X})}_{graph} = T$.

Proof. In view of Proposition 4.2, we only need to show that $T^{(\mathcal{X})}_{\text{graph}} = T$. An application of Theorem 3.1 shows that $\mathcal{X}$ is irreducible (as its MS presentation, $\mathcal{G}_\mathcal{X}$, is irreducible as a graph), so $T^{(\mathcal{X})}_{\text{graph}}$ can be defined. But now, Theorem 4.14 shows that T divides $T^{(\mathcal{X})}_{\text{graph}}$. In particular, $T^{(\mathcal{X})}_{\text{graph}} \geq T$.

On the other hand, the proof of Proposition 4.2 shows that $\text{per}(\mathcal{G}_\mathcal{X}) = T$. Thus, $T^{(\mathcal{X})}_{\text{graph}} \leq \text{per}(\mathcal{G}_\mathcal{X}) = T$. □

COROLLARY 4.19. *Let $\Sigma = \{0, 1\}$. For the PFT $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$, with T an arbitrary prime, and $\mathcal{F} = (\{11\}, \emptyset, \ldots, \emptyset)$, we have $T^{(\mathcal{X})}_{seq} = 1$ and $T^{(\mathcal{X})}_{graph} = T^{(\mathcal{X})}_{desc} = T$.*

As T in the proposition (and corollary) above can be an arbitrarily large prime, we see that it is possible for $T^{(\mathcal{X})}_{\text{seq}} \ll T^{(\mathcal{X})}_{\text{graph}}$. On the other hand, the following theorem provides a construction of a PFT $\mathcal{X}$ for which $T^{(\mathcal{X})}_{\text{seq}}$ can be arbitrarily larger than $T^{(\mathcal{X})}_{\text{graph}}$.

THEOREM 4.20. *Set $\Sigma = \{0, 1\}$ and $k \geq 2$, and let $\mathcal{P}$ denote the set of all periodic bi-infinite sequences over Σ with period $k!$. Consider the PFT $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},2\}}$ with $\mathcal{F} = (\mathcal{F}^{(0)}, \emptyset)$, such that $\mathcal{F}^{(0)} = \{w \in \Sigma^{2k!} : \exists \mathbf{x} \in \mathcal{P} \text{ such that } w \prec \mathbf{x}\}$. The following statements hold:*

(a) $\mathcal{X}$ is proper, and so, $T^{(\mathcal{X})}_{desc} = 2$;
(b) $\mathcal{X}$ is irreducible; and
(c) $T^{(\mathcal{X})}_{seq} \geq k + 1$ and $T^{(\mathcal{X})}_{graph} = 2$.

Proof. (a) Suppose $\mathcal{X}$ is not proper. Then there exists an SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ such that $\mathcal{X} = \mathcal{Y}$ and every forbidden word in $\mathcal{F}'$ has the same length ℓ.

Now consider two bi-infinite sequences $\mathbf{w} = (0^{k!}1)^\infty 0^{2k!}(10^{k!})^\infty$ and $\mathbf{w}' = (0^{k!}1)^\infty 0^{2k!}(10^{k!})^{2\ell}10^{2k!}(10^{k!})^\infty$. Observe that $\mathbf{w} \in \mathcal{X} = \mathcal{Y}$, and every subword of $\mathbf{w}'$ with length ℓ is a subword of $\mathbf{w}$ as well. Thus, $\mathbf{w}'$ must be in $\mathcal{Y}$. However, $\mathbf{w}' \notin \mathcal{X}$ since for the two integers i, j $(i < j)$ satisfying $0^{2k!} \prec_i \mathbf{w}'$ and $0^{2k!} \prec_j \mathbf{w}'$, we have $j - i = 2k! + 2\ell(k! + 1) + 1 \equiv 1 \pmod 2$, which cannot happen since $0^{2k!} \in \mathcal{F}^{(0)}$. Hence, $\mathcal{X}$ is proper.

(b) In this part of the proof, we use the following standard piece of notation: given a binary word $w = w_1 w_2 \ldots w_n$, we let $\overline{w}$ denote its complement $\overline{w_1}\,\overline{w_2} \ldots \overline{w_n}$, where $\overline{0} = 1$ and $\overline{1} = 0$.

The irreducibility of $\mathcal{X}$ can be proved, from Theorem 3.1, by showing that the SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}^{(0)}}$ is irreducible. This is because $\mathcal{B}_{2k!}(\mathcal{Y}) = \Sigma^{2k!} \setminus \mathcal{F}^{(0)}$ since for any $w' \in \Sigma^{2k!} \setminus \mathcal{F}^{(0)}$, $(w')^\infty \in \mathcal{Y}$, and furthermore, $\mathbf{x} = (0^{k!}1)^\infty \in \mathcal{Y}$, and $\mathbf{x}$ has period $k! + 1 \equiv 1 \pmod 2$. Also, to show that $\mathcal{Y}$ is irreducible, it is enough to check that two arbitrary words $u = u_1u_2\dots u_m, v = v_1v_2\dots v_n \in \mathcal{B}(\mathcal{Y})$ with lengths $m, n \ge 2k!$ can be concatenated through a word z so that $uzv \in \mathcal{B}(\mathcal{Y})$. In fact, it is enough to show that uzv contains no words from $\mathcal{F}^{(0)}$ as subwords. This is because if w is a word with length $n \ge 2k!$ containing no words from $\mathcal{F}^{(0)}$ as subwords, then $w \in \mathcal{B}(\mathcal{Y})$. Indeed, if we write $w = \alpha\gamma\beta$ with $|\alpha| = |\beta| = k!$, then it is easily verified that the bi-infinite sequence $(\overline{\alpha}\alpha)^\infty\gamma(\beta\overline{\beta})^\infty$ is in $\mathcal{Y}$.

Now take two arbitrary words $u = u_1u_2\dots u_m, v = v_1v_2\dots v_n \in \mathcal{B}(\mathcal{Y})$, where $m, n \ge 2k!$. For the words u and v, we can assume $\overline{v_1} = v_{k!+1} = u_{m-k!+1}$. (If not, replace v by $v' = \overline{v_{k!}}v$, which satisfies $\overline{v_1'} = v'_{k!+1}$. If it now happens that $\overline{v_1'} \ne u_{m-k!+1}$, then further replace v' by $v'' = \overline{v_1'}\,\overline{v_1}\dots\overline{v_{k!-1}}v' = v_{k!}\overline{v_1}\dots\overline{v_{k!}}v$ so that $\overline{v_1''} = v''_{k!+1} = u_{m-k!+1}$ holds. Clearly, the existence of a word z' so that $uz'v' \in \mathcal{B}(\mathcal{X})$ or $uz'v'' \in \mathcal{B}(\mathcal{X})$ shows the existence of a word z so that $uzv \in \mathcal{B}(\mathcal{X})$.)

For the concatenated word uv, the length-$2k!$ subwords of uv starting at u_i, $1 \le i \le m-2k!+1$, or v_j, $1 \le j \le n-2k!+1$, are clearly not in $\mathcal{F}^{(0)}$ since $u, v \in \mathcal{B}(\mathcal{Y})$. Furthermore, the length-$2k!$ subwords starting at u_i, $m-2k!+2 \le i \le m-k!+1$, are not in $\mathcal{F}^{(0)}$ since they contain $u_{m-k!+1}$ and v_1 satisfying $\overline{v}_1 = u_{m-k!+1}$, and similarly, neither are the length-$2k!$ subwords starting at u_i, $m-k!+2 \le i \le m$. Hence, uv contains no words in $\mathcal{F}^{(0)}$ as required.

(c) It is easy to see that $T_{\text{seq}}^{(\mathcal{X})} \ge k+1$. The definition of $\mathcal{X}$ implies that $\mathcal{X} \cap \mathcal{P} = \emptyset$. Since any bi-infinite sequence of period k or less also has $k!$ as a period, we conclude that $\mathcal{X}$ contains no sequences of period k or less.

As for $T_{\text{graph}}^{(\mathcal{X})}$, first observe that the proof of (b) in fact shows, via Theorem 3.1, that the MS presentation $\mathcal{G}_\mathcal{X}$ of $\mathcal{X}$ is irreducible. Therefore, if we can show that $\text{per}(\mathcal{G}_\mathcal{X}) = 2$, then we are done since $T_{\text{graph}}^{(\mathcal{X})} \ge 2$ from (a) and Theorem 4.14.

Consider the bi-infinite sequences $\mathbf{x} = (0^{k!}1)^\infty$ and $\mathbf{x} = (0^{k!}1^2)^\infty$. Both $\mathbf{x}$ and $\hat{\mathbf{x}}$ are in $\mathcal{X}$ as neither contains words in $\mathcal{F}^{(0)}$ as subwords. Therefore, for each subword u of $\mathbf{x}$ with length $2k!$, there exist in $\mathcal{G}_\mathcal{X}$ two states, $u^{(0)} \in \mathcal{V}^{(0)}$ and $u^{(1)} \in \mathcal{V}^{(1)}$, corresponding to u, and similarly, for each subword $\hat{u}$ of $\hat{\mathbf{x}}$ with length $2k!$, there exist two states, $\hat{u}^{(0)} \in \mathcal{V}^{(0)}$ and $\hat{u}^{(1)} \in \mathcal{V}^{(1)}$, corresponding to $\hat{u}$. Now let us focus on the state $u^{(0)} \in \mathcal{V}^{(0)}$ and $u^{(1)} \in \mathcal{V}^{(1)}$, where both $u^{(0)}$ and $u^{(1)}$ correspond to $0^{k!}10^{k!-1}$, a subword of $\mathbf{x}$. Since $0^{k!}10^{k!}10^{k!-1}$ is a subword of $\mathbf{x}$, there exists a path γ_{ij} generating $010^{k!-1}$ which starts at $u^{(i)}$ and terminates at $u^{(j)}$ for some $i, j \in \{0, 1\}$. However, the length of the path γ_{ij} is $k! + 1 \equiv 1 \pmod 2$. Therefore, from the structure of the MS presentation $\mathcal{G}_\mathcal{X}$, the starting state $u^{(i)}$ and the terminal state $u^{(j)}$ of γ_{ij} cannot be the same. It implies that $u^{(0)}$ is contained in the cycle with length $2(k!+1)$ generating $(010^{k!-1})^2$. Similarly, for the state $\hat{u}^{(0)} \in \mathcal{V}^{(0)}$ corresponding to $0^{k!}1^20^{k!-2}$, since $0^{k!}1^20^{k!}1^20^{k!-2}$ is a subword of $\hat{\mathbf{x}}$, there is a path (cycle) with length $k!+2$, generating $0^21^20^{k!-2}$, starting and terminating at $\hat{u}^{(0)} \in \mathcal{V}^{(0)}$.

Since $\mathcal{G}_\mathcal{X}$ is irreducible, $\text{per}(\mathcal{G}_\mathcal{X}) = \text{per}(u^{(0)}) = \text{per}(\hat{u}^{(0)})$. Thus, $\text{per}(\mathcal{G}_\mathcal{X})$ is a common divisor of $2(k!+1)$ and $k!+2$. As $k!+1$ and $k!+2$ are coprime, we see that $\text{per}(\mathcal{G}_\mathcal{X})$ divides 2. On the other hand, from (a) and Theorem 4.14, we have that $2|\text{per}(\mathcal{G}_\mathcal{X})$. Hence, $\text{per}(\mathcal{G}_\mathcal{X}) = 2$. □

5. Zeta Functions of the PFT's $\mathcal{X}_k$. The *zeta function* of a sofic shift $\mathcal{S}$ is a generating function for the number of period-n sequences in $\mathcal{S}$. To be precise, the zeta function $\zeta_{\mathcal{S}}(t)$ of a sofic shift $\mathcal{S}$ is defined to be

$$\zeta_{\mathcal{S}}(t) = \exp\left(\sum_{n=1}^{\infty} |P_n(\mathcal{S})|\, t^n/n\right), \tag{3}$$

where $P_n(\mathcal{S})$ is the set of periodic sequences in $\mathcal{S}$ with period n.

Theorem 4.7 shows that periodic sequences in the PFT's $\mathcal{X}_k$ (as defined in Section 4.1) can only have periods that are multiples of $2^{\lceil \log_2 k \rceil}$. Thus, for any $n \not\equiv 0 \pmod{2^{\lceil \log_2 k \rceil}}$, the number of period-$n$ sequences in $\mathcal{X}_k$ is zero. It turns out that the number of periodic sequences in $\mathcal{X}_k$ with a period $n \equiv 0 \pmod{2^{\lceil \log_2 k \rceil}}$ can actually be counted by a direct argument, thus allowing us to compute, via (3), the zeta function of $\mathcal{X}_k$.

The count of periodic sequences in $\mathcal{X}_k$ is particularly simple in the case when k is a power of 2, which is what we present here. The case when k is not a power of 2 can also be handled by similar arguments, but that requires a little more effort.

Theorem 5.1. *Let $j \geq 0$, and consider the PFT $\mathcal{X}_k$ with $k = 2^j$. For the set, $P_n(\mathcal{X}_k)$, of periodic sequences in $\mathcal{X}_k$ with period n, we have*

$$|P_n(\mathcal{X}_1)| = \begin{cases} 2^{1+n/2} - 1 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

and when $k = 2^j \geq 2$,

$$|P_n(\mathcal{X}_k)| = \begin{cases} 2^{k-1}\,(2^{(n-k+2)/2} - 1) & \text{if } \ n \equiv 0 \pmod{k} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the zeta function $\zeta_{\mathcal{X}_k}(t)$ of $\mathcal{X}_k$ with $k = 2^j$ is given by

$$\zeta_{\mathcal{X}_k}(t) = \begin{cases} \dfrac{1+t}{1-2t^2} & \text{when } k = 1 \\ \dfrac{(1-t^k)^{k^{-1}2^{k-1}}}{\left(1-(2t^2)^{k/2}\right)^{k^{-1}\,2^{k/2}}} & \text{otherwise} \end{cases}$$

Proof. Throughout this proof, we call a bi-infinite sequence $\mathbf{x}$ (resp. a word w) *normal* with respect to (wrt) $\mathcal{X}_k$ if $f \not\prec_i \mathbf{x}$ (resp. if $f \not\prec_i w$) for any $f \in \mathcal{F}_k^{(0)}$ and even integer i. Let $N_n(\mathcal{X}_k)$ denote the set of periodic sequences $\mathbf{x}$ in $P_n(\mathcal{X}_k)$ such that $\mathbf{x}$ is normal wrt $\mathcal{X}_k$, and similarly, let $M_n(\mathcal{X}_k)$ denote the set of periodic sequences $\mathbf{x}$ in $P_n(\mathcal{X}_k)$ such that $\sigma(\mathbf{x})$ is normal wrt $\mathcal{X}_k$. Then, we have $P_n(\mathcal{X}_k) = N_n(\mathcal{X}_k) \cup M_n(\mathcal{X}_k)$, so $|P_n(\mathcal{X}_k)|$ is given by $|N_n(\mathcal{X}_k)| + |M_n(\mathcal{X}_k)| - |N_n(\mathcal{X}_k) \cap M_n(\mathcal{X}_k)|$. Observe that $|N_n(\mathcal{X}_k)| = |M_n(\mathcal{X}_k)|$ since the map $\sigma|_{N_n(\mathcal{X}_k)} : N_n(\mathcal{X}_k) \longrightarrow M_n(\mathcal{X}_k)$, which is the shift map on $N_n(\mathcal{X}_k)$, is bijective. Also, for $\mathbf{x} \in P_n(\mathcal{X}_k)$, it follows easily from the definitions that $\mathbf{x} \in N_n(\mathcal{X}_k) \cap M_n(\mathcal{X}_k)$ iff $f \not\prec \mathbf{x}$ for any $f \in \mathcal{F}_k^{(0)}$.

First we focus on $\mathcal{X}_1$. Recall that $\mathcal{F}_1 = (\{0\}, \emptyset)$. When n is even, observe that $\mathbf{x} = (x_0 x_1 \ldots x_{n-1})^\infty \in N_n(\mathcal{X}_1)$ if and only if $x_i = 1$ for any even integer i, $0 \leq i \leq n-1$ (but $x_{i'}$ can be 0 or 1 for any odd integer i', $0 \leq i' \leq n-1$). Since n is even, the number of odd integers between 0 and $n-1$ is $n/2$, so $|N_n(\mathcal{X}_1)| = 2^{n/2}$, and hence, $|M_n(\mathcal{X}_1)| = 2^{n/2}$ as well. Also, 1^∞ is the only periodic sequence in

$N_n(\mathcal{X}_1) \cap M_n(\mathcal{X}_1)$. Hence, $|P_n(\mathcal{X}_1)| = 2 \times 2^{2/n} - 1$. On the other hand, when n is odd, $\mathbf{x} = (x_0 x_1 \ldots x_{n-1})^\infty \in P_n(\mathcal{X}_1)$ iff $\mathbf{x} = 1^\infty$, since if there is an integer i, $0 \le i \le n-1$, such that $x_i = 0$, then $x_i = x_{i+n} = 0$ and exactly one of i and $i+n$ is even. Hence, $|P_n(\mathcal{X}_1)| = 1$.

Next consider $\mathcal{X}_k$ when $k = 2^j$ for some $j \ge 1$. Clearly, $|P_n(\mathcal{X}_k)| = 0$ when $n \not\equiv 0 \pmod{k}$ from Theorem 4.7, so suppose $n = tk$ for some $t \ge 1$. Recall from Proposition 4.4 that for each $w = w_1 w_2 \ldots w_k \in \Sigma^k$, $\psi^{k-1}(w)$ is given by $w_1 \oplus w_2 \oplus \cdots \oplus w_k$, and therefore, $f = f_1 f_2 \ldots f_k \in \mathcal{F}_k^{(0)}$ iff $f_1 \oplus f_2 \oplus \cdots \oplus f_k = 0$.

We first count the number of bi-infinite sequences in $N_{tk}(\mathcal{X}_k)$. Note that if $\mathbf{x} = (x_0 x_1 \ldots x_{tk-1})^\infty$ is in $N_{tk}(\mathcal{X}_k)$, then clearly, $x_0 x_1 \ldots x_{tk-1}$ must be normal wrt $\mathcal{X}_k$. The converse is also true, as we now show. Suppose that a word $x = x_0 x_1 \ldots x_{tk-1} \in \Sigma^{tk}$ is normal wrt $\mathcal{X}_k$, *i.e.*, $\oplus_{m=i}^{i+k-1} x_m = 1$ for any even integer $0 \le i \le (t-1)k$. Then, $\oplus_{m=hk}^{(h+1)k-1} x_m = \oplus_{m=hk+2}^{(h+1)k+1} x_m = 1$ for any $0 \le h \le t-2$, so we have $x_0 \oplus x_1 = x_k \oplus x_{k+1} = \cdots = x_{(t-1)k} \oplus x_{(t-1)k+1}$. Hence, $x_{(t-1)k+2} \oplus \cdots \oplus x_{tk-1} \oplus x_0 \oplus x_1 = x_{(t-1)k+2} \oplus \cdots \oplus x_{tk-1} \oplus x_{(t-1)k} \oplus x_{(t-1)k+1} = 1$. Therefore, the word $x' = x'_0 x'_1 \ldots x'_{tk-1} = x_2 x_3 \ldots x_{tk-1} x_0 x_1$ is also normal wrt $\mathcal{X}_k$. In other words, if $x = x_0 x_1 \ldots x_{tk-1} \in \Sigma^{tk}$ is normal wrt $\mathcal{X}_k$, then so is $x_2 x_3 \ldots x_{tk-1} x_0 x_1$. This implies that the periodic bi-infinite sequence $\mathbf{x} = x^\infty$ is normal wrt $\mathcal{X}_k$, and thus, $\mathbf{x} \in N_{tk}(\mathcal{X}_k)$.

Thus, $|N_{tk}(\mathcal{X}_k)|$ is equal to the number of words $x_0 x_1 \ldots x_{tk-1} \in \Sigma^{tk}$ that are normal wrt $\mathcal{X}_k$. We can generate all such words $x_0 x_1 \ldots x_{tk-1}$ as follows.

1. Choose $x_0 x_1 \ldots x_{k-1}$ so that $x_0 x_1 \ldots x_{k-1} \notin \mathcal{F}_k^{(0)}$. We have $|\Sigma^k \backslash \mathcal{F}_k^{(0)}| = 2^{k-1}$ choices for $x_0 x_1 \ldots x_{k-1}$.
2. As for $x_k x_{k+1} \ldots x_{tk-1}$, there is a unique choice for x_i, for any even $i \in \{k, k+1, \ldots, tk-1\}$, such that $x_{i-k+1} x_{i-k+2} \ldots x_i \notin \mathcal{F}_k^{(0)}$ (*i.e.*, $\oplus_{m=i-k+1}^{i} x_m = 1$); but when i is odd, there are no restrictions on x_i. Since there are $(tk-k)/2$ odd numbers between k and $tk-1$, we have $2^{(tk-k)/2}$ choices for $x_k x_{k+1} \ldots x_{tk-1}$. (This count also holds for $t = 1$, since $x_k x_{k+1} \ldots x_{tk-1} = \epsilon$ in this case.)

Therefore, we have in total $2^{k-1} 2^{(tk-k)/2}$ words $x_0 x_1 \ldots x_{tk-1}$ that are normal wrt $\mathcal{X}_k$, and hence, $|N_{tk}(\mathcal{X}_k)| = |M_{tk}(\mathcal{X}_k)| = 2^{k-1} 2^{(tk-k)/2}$.

Finally, we count the number of bi-infinite sequences that are in $N_{tk}(\mathcal{X}_k) \cap M_{tk}(\mathcal{X}_k)$. These are the sequences $\mathbf{x}$ in $P_{tk}(\mathcal{X}_k)$ such that $f \not\prec \mathbf{x}$ for all $f \in \mathcal{F}_k^{(0)}$. An argument similar to that made above for sequences in $N_{tk}(\mathcal{X}_k)$ shows that $\mathbf{x} = (x_0 x_1 \ldots x_{tk-1})^\infty \in P_{tk}(\mathcal{X}_k)$ contains no words from $\mathcal{F}_k^{(0)}$ as subwords iff $x_0 x_1 \ldots x_{tk-1}$ contains no words from $\mathcal{F}_k^{(0)}$ as subwords. To construct $x_0 x_1 \ldots x_{tk-1} \in \Sigma^{tk}$ that does not contain any word in $\mathcal{F}_k^{(0)}$ as a subword, we only have the freedom to choose $x_0 x_1 \ldots x_{k-1}$ to be a word in $\Sigma^k \setminus \mathcal{F}_k^{(0)}$. Once this is chosen, all other x_i's $(k \le i \le tk-1)$ are uniquely determined. Indeed, each such x_i must be chosen so that $\oplus_{m=i-k+1}^{i} x_m = 1$. It follows that $|N_{tk}(\mathcal{X}_k) \cap M_{tk}(\mathcal{X}_k)| = |\Sigma^k \setminus \mathcal{F}_k^{(0)}| = 2^{k-1}$.

Therefore,

$$\begin{aligned} |P_{tk}(\mathcal{X}_k)| &= |N_{tk}(\mathcal{X}_k)| + |M_{tk}(\mathcal{X}_k)| - |N_{tk}(\mathcal{X}_k) \cap M_{tk}(\mathcal{X}_k)| \\ &= 2(2^{k-1} 2^{(tk-k)/2}) - 2^{k-1} \quad = \quad 2^{k-1}(2^{(tk-k+2)/2} - 1), \end{aligned}$$

which is the stated formula for $|P_n(\mathcal{X}_k)|$ when $n \equiv 0 \pmod{k}$.

The expression for $\zeta_{\mathcal{X}_k}(t)$ can now be obtained from (3) by a simple calculation. □

The above proof also yields an expression for the zeta function of the SFT $\mathcal{Y}_k$, $k = 2^j$, with forbidden set $\mathcal{F}' = \mathcal{F}_k^{(0)}$. Observe that the set $P_n(\mathcal{Y}_k)$ of periodic sequences in $\mathcal{Y}_k$ with period n is equal to $N_n(\mathcal{X}_k) \cap M_n(\mathcal{X}_k)$. We thus have the following corollary.

Corollary 5.2. *Let $k = 2^j$, $j \geq 0$, and consider the SFT $\mathcal{Y}_k$ with forbidden set $\mathcal{F}' = \psi^{-(k-1)}(\{0\})$. For the set $P_n(\mathcal{Y}_k)$ of periodic sequences in $\mathcal{Y}_k$ with period n, we have*

$$|P_n(\mathcal{Y}_k)| = \begin{cases} 2^{k-1} & \text{if } n \equiv 0 \pmod{k} \\ 0 & \text{otherwise} \end{cases}$$

Hence, the zeta function $\zeta_{\mathcal{Y}_k}(t)$ of $\mathcal{Y}_k$ is given by

$$\zeta_{\mathcal{Y}_k}(t) = \frac{1}{(1-t^k)^{k^{-1}2^{k-1}}}.$$

As remarked at the beginning of this section, arguments similar to those used in the proof of Theorem 5.1 can also be used to count the number of period-n sequences in $\mathcal{X}_k$ in the case when k is not a power of 2. However, in this case, it requires a little more effort to show that for a word $x_0x_1\ldots x_{n-1} \in \Sigma^n$ with $n \equiv 0 \pmod{2^{\lceil \log_2 k \rceil}}$, we have $\mathbf{x} = (x_0x_1\ldots x_{n-1})^\infty \in N_n(\mathcal{X}_k)$ iff $x_0x_1\ldots x_{n-1}$ is normal wrt $\mathcal{X}_k$, and that $\mathbf{x} = (x_0x_1\ldots x_{n-1})^\infty \in N_n(\mathcal{X}_k) \cap M_n(\mathcal{X}_k)$ iff $x_0x_1\ldots x_{n-1}$ contains no words from $\mathcal{F}_k^{(0)}$ as subwords. For completeness, we offer without proof the expressions for $|P_n(\mathcal{X}_k)|$ for arbitrary $k \geq 2$ (k not necessarily a power of 2). For even values of $k \geq 2$, we have

$$|P_n(\mathcal{X}_k)| = \begin{cases} 2^{k-1}\,(2^{(n-k+2)/2} - 1) & \text{if } n \equiv 0 \pmod{2^{\lceil \log_2 k \rceil}} \\ 0 & \text{otherwise,} \end{cases}$$

and for odd values of $k \geq 2$, we have

$$|P_n(\mathcal{X}_k)| = \begin{cases} 2^{k-1}\,(2^{(n-k+3)/2} - 1) & \text{if } n \equiv 0 \pmod{2^{\lceil \log_2 k \rceil}} \\ 0 & \text{otherwise.} \end{cases}$$

From the above, we can obtain expressions for the zeta functions of the $\mathcal{X}_k$'s. For $k \geq 2$, setting $K = 2^{\lceil \log_2 k \rceil}$, we have

$$\zeta_{\mathcal{X}_k}(t) = \begin{cases} \dfrac{(1-t^K)^{K^{-1}2^{k-1}}}{\left(1-(2t^2)^{K/2}\right)^{K^{-1}\,2^{k/2}}} & \text{if } k \text{ is even} \\[2ex] \dfrac{(1-t^K)^{K^{-1}2^{k-1}}}{\left(1-(2t^2)^{K/2}\right)^{K^{-1}\,2^{(k+1)/2}}} & \text{if } k \text{ is odd.} \end{cases}$$

These expressions could be useful as test cases for verifying the correctness of a general formula for the zeta function of a PFT.

Appendix A. We prove here that the PFT $\mathcal{X}_k$ is irreducible iff $1 \leq k \leq 6$. For this, we will need to develop an understanding of the structure of the MS presentation, $\mathcal{G}_{\mathcal{X}_k}$, of $\mathcal{X}_k$, constructed by means of the MS algorithm described in Section 2.

For $b \in \{0,1\}$, let $\overline{b}$ denote the complement of b, *i.e.*, $\overline{0} = 1$ and $\overline{1} = 0$. Observe that $f = f_1 f_2 \dots f_{k-1} f_k \in \mathcal{F}_k^{(0)}$ iff $f' = f_1 f_2 \dots f_{k-1} \overline{f}_k \notin \mathcal{F}_k^{(0)}$ and $f'' = \overline{f}_1 f_2 \dots f_{k-1} f_k \notin \mathcal{F}_k^{(0)}$, as $\psi^{k-1}(f') \neq \psi^{k-1}(f)$ and $\psi^{k-1}(f'') \neq \psi^{k-1}(f)$. From this, we can infer some of the structure of the graph $\mathcal{G}'$ constructed in Step 4 of the MS algorithm. The graph $\mathcal{G}'$ is a directed bipartite graph on a state set that is the disjoint union of $\mathcal{V}^{(0)} = \Sigma^k \setminus \mathcal{F}_k^{(0)}$ and $\mathcal{V}^{(1)} = \Sigma^k$. Each state in $\mathcal{V}^{(1)}$ has exactly one outgoing edge, which terminates in some state in $\mathcal{V}^{(0)}$. Similarly, each state in $\mathcal{V}^{(1)}$ has exactly one incoming edge, which originates in some state in $\mathcal{V}^{(0)}$. On the other hand, each state in $\mathcal{V}^{(0)}$ has two outgoing edges, one going to a state in $\mathcal{F}_k^{(0)} \subset \mathcal{V}^{(1)}$, and the other going to a state in $\Sigma^k \setminus \mathcal{F}_k^{(0)} \subset \mathcal{V}^{(1)}$. And finally, each state in $\mathcal{V}^{(0)}$ has two incoming edges, one from a state in $\mathcal{F}_k^{(0)} \subset \mathcal{V}^{(1)}$, and the other from a state in $\Sigma^k \setminus \mathcal{F}_k^{(0)} \subset \mathcal{V}^{(1)}$.

Since every state in $\mathcal{G}'$ has incoming and outgoing edges, we see from Step 5 of the MS algorithm that $\mathcal{G}_{\mathcal{X}_k} = \mathcal{G}'$. Noting further that each state in $\mathcal{V}^{(1)} = \Sigma^k$ has a path α of length k terminating at it, every word in Σ^k must be generated within a bi-infinite path in $\mathcal{G}_{\mathcal{X}_k}$ (see Remark 2.1). Hence, $\Sigma^k \subset \mathcal{B}(\mathcal{X}_k)$.

We need one more observation. The graph $\mathcal{G}_{\mathcal{X}_k}$ is irreducible iff, for each pair of states V, V' in $\mathcal{F}_k^{(0)} \subset \mathcal{V}^{(1)}$, there is a path in $\mathcal{G}_{\mathcal{X}_k}$ starting at V and terminating at V'. This is because for any state $U \notin \mathcal{F}_k^{(0)}$, there exists a path starting at U and terminating at a state in $\mathcal{F}_k^{(0)}$, and another path starting at a state in $\mathcal{F}_k^{(0)}$ and terminating at U.

Lemma A.1. *For any $k \geq 1$, $\mathcal{X}_k$ is irreducible iff its MS presentation $\mathcal{G}_{\mathcal{X}_k}$ is irreducible.*

Proof. Clearly, $\mathcal{X}_k$ is irreducible when $\mathcal{G}_{\mathcal{X}_k}$ is irreducible, by definition. Conversely, suppose $\mathcal{G}_{\mathcal{X}_k}$ is not irreducible. Then, as noted above, there must be a pair of states $V, V' \in \mathcal{F}_k^{(0)}$ such that there is no path in $\mathcal{G}_{\mathcal{X}_k}$ that starts at V and terminates at V'. This implies that there is no word z so that $VzV' \in \mathcal{B}(\mathcal{X}_k)$. However, both V and V' are words in $\mathcal{B}(\mathcal{X}_k)$. Hence, $\mathcal{X}_k$ is not irreducible. □

Proposition A.2. *$\mathcal{X}_k$ is irreducible iff $1 \leq k \leq 6$.*

Proof. Firstly, $\mathcal{X}_k$ is irreducible for $1 \leq k \leq 6$ since its MS presentation may be verified to be irreducible as a graph.

To show that $\mathcal{X}_k$ is not irreducible when $k \geq 7$, it is enough to prove that $\mathcal{X}_7$ is not irreducible. This is because $\mathcal{X}_{k+1} = \psi^{-1}(\mathcal{X}_k)$ by Proposition 4.6(a), from which we easily obtain that if $\mathcal{X}_{k+1}$ is irreducible, then so must be $\mathcal{X}_k = \psi(\mathcal{X}_{k+1})$. However, it may be directly verified by explicit construction that the MS presentation, $\mathcal{G}_{\mathcal{X}_7}$, of $\mathcal{X}_7$ is not irreducible. Hence, from Lemma A.1, we conclude that $\mathcal{X}_7$ is not irreducible. □

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