On Codes of Bounded Trellis Complexity*

Navin Kashyap
Dept. Mathematics & Statistics
Queen's University
Kingston, ON, K7L 2H4, Canada
Email: nkashyap@mast.queensu.ca

***Abstract*— In this paper, we initiate a structure theory of linear codes with bounded trellis complexity. The theory is based on the observation that the family of linear codes over $\mathbb{F}_q$, some permutation of which has trellis state-complexity at most w, is a minor-closed family. It then follows from a deep result of matroid theory that such codes are characterized by finitely many excluded minors. We provide the complete list of excluded minors for $w = 1$, and give a partial list for $w = 2$. We also give a polynomial-time algorithm for determining whether or nor a given code has a permutation with state-complexity at most 1.**

I. Introduction

Given a linear code $\mathcal{C}$ over the finite field $\mathbb{F}_q$, the fundamental problem of trellis decoding is to find an equivalent code $\mathcal{C}'$ whose minimal trellis representation has the least state-complexity among all codes equivalent to $\mathcal{C}$. This problem is known to be difficult; indeed, for any fixed finite field $\mathbb{F}_q$, the following decision problem is NP-complete [12]:

Problem: Strong Trellis State-Complexity (STSC)
Instance: An $m \times n$ generator matrix for a linear code $\mathcal{C}$ over $\mathbb{F}_q$, and an integer $w > 0$.
Question: Is there a code $\mathcal{C}'$ equivalent to $\mathcal{C}$, whose minimal trellis has state-complexity at most w?

The outlook is not so gloomy if we weaken the above problem by *not* considering the integer w to be a part of the input to the problem. In other words, for a *fixed* finite field $\mathbb{F}_q$, and a *fixed* integer $w > 0$, we consider the following problem:

Problem: Weak Trellis State-Complexity (WTSC)
Instance: An $m \times n$ generator matrix for a linear code $\mathcal{C}$ over $\mathbb{F}_q$.
Question: Is there a code $\mathcal{C}'$ equivalent to $\mathcal{C}$, whose minimal trellis has state-complexity at most w?

In this paper, we provide strong evidence in support of our belief that WTSC is solvable in polynomial time.

Our approach to the above problem relies on the notion of code minors. A *minor* of a code $\mathcal{C}$ is any code that can be obtained from $\mathcal{C}$ by a (possibly empty) sequence of shortening and puncturing operations. A minor of $\mathcal{C}$ that is not $\mathcal{C}$ itself is called a *proper minor* of $\mathcal{C}$. A family, $\mathfrak{F}$, of codes over $\mathbb{F}_q$ is said to be *minor-closed* if, for each $\mathcal{C} \in \mathfrak{F}$, any code equivalent to a minor of $\mathcal{C}$ is also in $\mathfrak{F}$. A code, $\mathcal{D}$, is said to be an *excluded minor* for a minor-closed family $\mathfrak{F}$, if $\mathcal{D} \notin \mathfrak{F}$, but every proper minor of $\mathcal{D}$ is in $\mathfrak{F}$. It is not hard to see that, if $\mathfrak{F}$ is a minor-closed family, then a code $\mathcal{C}$ is in $\mathfrak{F}$ iff no minor of $\mathcal{C}$ is an excluded minor of $\mathfrak{F}$.

Given an integer $w > 0$, we define a family, $TC_{w;q}$, of linear codes over $\mathbb{F}_q$ as follows: a code $\mathcal{C}$ is in $TC_{w;q}$ iff there exists a code $\mathcal{C}'$ equivalent to $\mathcal{C}$, such that the minimal trellis of $\mathcal{C}'$ has state-complexity at most w. It is fairly straightforward to show, as we do in Section III, that $TC_{w;q}$ is minor-closed. A deep result of matroid theory [7, Theorem 1.4] then implies that $TC_{w;q}$ in fact has a *finite* number of excluded minors. This connection to matroid theory is drawn in Section IV.

The problem of deciding whether or not $\mathcal{C}$ is in $TC_{w;q}$ is thus equivalent to that of deciding, for each of the finitely many excluded minors, $\mathcal{D}$, of $TC_{w;q}$, whether or not $\mathcal{C}$ contains $\mathcal{D}$ as a minor. Now, a conjecture in matroid theory [6, Conjecture 1.3], when extended to codes, states that if $\mathcal{D}$ is a fixed code over $\mathbb{F}_q$, then, given any code $\mathcal{C}$ over $\mathbb{F}_q$, it is decidable in time polynomial in the length of $\mathcal{C}$, whether or not $\mathcal{C}$ contains $\mathcal{D}$ as a minor. Hence, if the conjecture is true[1], then membership of a code $\mathcal{C}$ in $TC_{w;q}$ can be decided in time polynomial in the length of $\mathcal{C}$. There is firm evidence in the literature in support of the conjecture [6].

In Sections V and VI, we find an explicit list of excluded minors for the binary code families $TC_{1;2}$ and $TC_{2;2}$ (which we denote simply by TC_1 and TC_2, respectively). We show that our list of excluded minors for TC_1 is complete, but we do not make the same claim for TC_2. We also give a polynomial-time algorithm for deciding membership of a code in TC_1.

Excluded-minor characterizations of $TC_{w;q}$ are a means of identifying those substructures within a code that prevent the code from having low trellis state-complexity. These characterizations could thus be used to design codes of low trellis state-complexity. It should also be pointed out that $TC_{w;q}$ is not an asymptotically good code family, in the sense that either the dimension or the minimum distance of any sequence of codes from this family must grow sub-linearly with codelength [13]. Thus, its excluded-minor characterizations tell us what kind of code substructures must be present in asymptotically good code families.

We begin our exposition by defining the notation to be used in the rest of the paper.

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[1] Actually, the conjecture as stated here for codes is slightly stronger than the original conjecture for matroids. However, the conclusion about $TC_{w;q}$ holds even if the weaker matroid conjecture turns out to be true.

II. Notation

Let $\mathcal{C}$ be a linear code of length n over $\mathbb{F}_q$. The dimension of $\mathcal{C}$ is denoted by $\dim(\mathcal{C})$, and the coordinates of $\mathcal{C}$ are indexed by the integers from the set $[n] = \{1, 2, \ldots, n\}$ as usual. We will also associate with the coordinates of $\mathcal{C}$ a set, $E(\mathcal{C})$, of *coordinate labels*, so that there is a bijection $\alpha_{\mathcal{C}} : [n] \to E(\mathcal{C})$. The *label sequence* of $\mathcal{C}$ is defined to be the n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i = \alpha_{\mathcal{C}}(i)$. From now on, we will simply let $\alpha_{\mathcal{C}}$ denote the label sequence of $\mathcal{C}$. Unless specified otherwise (as in the case of code minors and duals below), we will, by default, set $E(\mathcal{C})$ to be $[n]$, and $\alpha_{\mathcal{C}}$ to be the n-tuple $(1, 2, 3, \ldots, n)$. In such a case, the label of each coordinate is the same as its index.

Given a $J \subset E(\mathcal{C})$, we will denote by $\mathcal{C}/J$ (resp. $\mathcal{C} \setminus J$) the code obtained from $\mathcal{C}$ by puncturing (resp. shortening at) those coordinates having labels in J. Thus, $\mathcal{C} \setminus J = (\mathcal{C}^\perp/J)^\perp$. We will also sometimes use $\mathcal{C}|_J$ to denote the restriction of $\mathcal{C}$ to the coordinates with labels in J, *i.e.*, $\mathcal{C}|_J = \mathcal{C}/J^c$, where J^c denotes the set difference $E(\mathcal{C}) - J$.

It is straightforward to see that for any $X \subset E(\mathcal{C})$, we have

$$\dim(\mathcal{C} \setminus X) = \dim(\mathcal{C}) - \dim(\mathcal{C}|_X). \tag{1}$$

Also, if $X, J \subset E(\mathcal{C})$ are disjoint, then we obviously have

$$\dim((\mathcal{C}/X)|_J) = \dim(\mathcal{C}|_J), \tag{2}$$

and it is not difficult to deduce from (1) that

$$\dim((\mathcal{C} \setminus X)|_J) = \dim(\mathcal{C}|_{J \cup X}) - \dim(\mathcal{C}|_X). \tag{3}$$

Another property of dimension that is often useful is the property of submodularity: for any $X, Y \subset E(\mathcal{C})$,

$$\dim(\mathcal{C}|_{X \cup Y}) + \dim(\mathcal{C}|_{X \cap Y}) \le \dim(\mathcal{C}|_X) + \dim(\mathcal{C}|_Y). \tag{4}$$

A minor of $\mathcal{C}$ is a code of the form $\mathcal{C}/X \setminus Y$ for disjoint subsets $X, Y \subset E(\mathcal{C})$. We set $E(\mathcal{C}/X \setminus Y) = E(\mathcal{C}) - (X \cup Y)$, and take the label sequence of $\mathcal{C}/X \setminus Y$ to be the $(n - |X \cup Y|)$-tuple obtained from the n-tuple $\alpha_{\mathcal{C}} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ by simply removing those entries that are in $X \cup Y$.

The label sequence of the dual code $\mathcal{C}^\perp$ is specified to be the same as that of $\mathcal{C}$, *i.e.*, $\alpha_{\mathcal{C}^\perp} = \alpha_{\mathcal{C}}$. Thus, in particular, $E(\mathcal{C}) = E(\mathcal{C}^\perp)$.

Two length-n linear codes $\mathcal{C}$ and $\mathcal{C}'$ over $\mathbb{F}_q$ are defined to be *equivalent* if there is an $n \times n$ permutation matrix Π and an invertible $n \times n$ diagonal matrix Δ, such that $\mathcal{C}'$ is the image of $\mathcal{C}$ under the vector space isomorphism $\phi : \mathbb{F}_q^n \to \mathbb{F}_q^n$ defined by $\phi(\mathbf{x}) = (\Pi\Delta)\mathbf{x}$. Informally, $\mathcal{C}'$ is equivalent to $\mathcal{C}$ if $\mathcal{C}'$ can be obtained by first multiplying the coordinates of $\mathcal{C}$ by some nonzero elements of $\mathbb{F}_q$, and then applying a coordinate permutation. In such a case, we will write $\mathcal{C} \equiv \mathcal{C}'$. The equivalence class of codes equivalent to $\mathcal{C}$ will be denoted by $[\mathcal{C}]$.

III. Trellis Complexity and Minors

We will define trellis state-complexity via the rather useful notion of the *connectivity function* of a linear code $\mathcal{C}$. This is the function $\lambda_{\mathcal{C}} : 2^{E(\mathcal{C})} \to \mathbb{Z}$ defined by

$$\lambda_{\mathcal{C}}(J) = \dim(\mathcal{C}|_J) + \dim(\mathcal{C}|_{J^c}) - \dim(\mathcal{C}), \tag{5}$$

for each $J \subset E(\mathcal{C})$. It is obvious that for any $J \subset E(\mathcal{C})$, we have $\lambda_{\mathcal{C}}(J) \ge 0$ and $\lambda_{\mathcal{C}}(J) = \lambda_{\mathcal{C}}(J^c)$. Observe also that $\lambda_{\mathcal{C}}(\emptyset) = \lambda_{\mathcal{C}}(E(\mathcal{C})) = 0$.

Let $\mathcal{C}$ be a linear code over $\mathbb{F}_q$, with label sequence $\alpha_{\mathcal{C}} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The *state-complexity profile* [5], [9] of $\mathcal{C}$ is the sequence $\mathbf{s}(\mathcal{C}) = (s_0(\mathcal{C}), s_1(\mathcal{C}), \ldots, s_n(\mathcal{C}))$ defined as follows: $s_0(\mathcal{C}) = s_n(\mathcal{C}) = 0$, and for $1 \le i \le n-1$,

$$s_i(\mathcal{C}) = \lambda_{\mathcal{C}}(\{\alpha_1, \alpha_2, \ldots, \alpha_i\}). \tag{6}$$

The quantities $s_i(\mathcal{C})$ determine the size of the minimal trellis of $\mathcal{C}$ — the number of vertices (states) at time i in the minimal trellis is precisely $q^{s_i(\mathcal{C})}$. The *state-complexity* of (the minimal trellis of) the code $\mathcal{C}$ is defined to be $s_{\max}(\mathcal{C}) = \max_{i \in [n]} s_i(\mathcal{C})$.

As was noted by Muder [15], equivalent codes $\mathcal{C}$ and $\mathcal{C}'$ may have very different minimal trellises. Therefore, we may have $\mathbf{s}(\mathcal{C}) \ne \mathbf{s}(\mathcal{C}')$, and even $s_{\max}(\mathcal{C}) \ne s_{\max}(\mathcal{C}')$. It thus makes sense to consider the *minimum state-complexity*,

$$\sigma[\mathcal{C}] = \min_{\mathcal{C}' \in [\mathcal{C}]} s_{\max}(\mathcal{C}') = \min_{\mathcal{C}' \in [\mathcal{C}]} \max_{i \in [n]} s_i(\mathcal{C}'),$$

of codes within the equivalence class $[\mathcal{C}]$.

In this paper, we are primarily concerned with the family, $TC_{w;q}$, of codes $\mathcal{C}$ over $\mathbb{F}_q$ that satisfy $\sigma[\mathcal{C}] \le w$, where w is a fixed positive integer. Clearly, $TC_{w;q}$ is closed under code equivalence, since for any $\mathcal{C} \in TC_{w;q}$, we have $[\mathcal{C}] \subset TC_{w;q}$. Furthermore, $TC_{w;q}$ is closed under duality — $\mathcal{C} \in TC_{w;q}$ iff $\mathcal{C}^\perp \in TC_{w;q}$. This follows from the well known fact [4] that $\mathbf{s}(\mathcal{C}) = \mathbf{s}(\mathcal{C}^\perp)$ for any code $\mathcal{C}$, and hence, $\sigma[\mathcal{C}] = \sigma[\mathcal{C}^\perp]$.

It is also easily verified that $TC_{w;q}$ is closed under direct sums, *i.e.*, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are in $TC_{w;q}$, then so is $\mathcal{C}_1 \oplus \mathcal{C}_2$. This is because $s_{\max}(\mathcal{C}_1 \oplus \mathcal{C}_2) = \max\{\, s_{\max}(\mathcal{C}_1), s_{\max}(\mathcal{C}_2) \,\}$, so that $\sigma[\mathcal{C}_1 \oplus \mathcal{C}_2] \le \max\{\sigma[\mathcal{C}_1], \sigma[\mathcal{C}_2]\}$.

The main aim of this section is to show that $TC_{w;q}$ is a minor-closed family, *i.e.*, if $\mathcal{C} \in TC_{w;q}$, then for any minor, $\mathcal{D}$, of $\mathcal{C}$, $[\mathcal{D}] \subset TC_{w;q}$. For this, we need to study the connectivity function $\lambda_{\mathcal{C}}$ in more detail.

From the fact that $\mathbf{s}(\mathcal{C}') = \mathbf{s}(\mathcal{C}'^\perp)$ for all codes $\mathcal{C}'$ in the equivalence class $[\mathcal{C}]$, we see that $\lambda_{\mathcal{C}}(J) = \lambda_{\mathcal{C}^\perp}(J)$ for all $J \subset E(\mathcal{C})$. Moreover, it follows easily from (4) that $\lambda_{\mathcal{C}}$ also has the submodularity property: for any $X, Y \subset E(\mathcal{C})$,

$$\lambda_{\mathcal{C}}(X \cup Y) + \lambda_{\mathcal{C}}(X \cap Y) \le \lambda_{\mathcal{C}}(X) + \lambda_{\mathcal{C}}(Y). \tag{7}$$

The next result, which is less obvious, is well known in the matroid theory literature. We provide a proof here for the sake of completeness.

Lemma III.1 *If $\mathcal{D}$ is a minor of $\mathcal{C}$, then for all $J \subset E(\mathcal{D})$, $\lambda_{\mathcal{D}}(J) \le \lambda_{\mathcal{C}}(J)$.*

Proof: We start by noting that we may write $\lambda_{\mathcal{C}}(J)$ as

$$\lambda_{\mathcal{C}}(J) = \dim(\mathcal{C}|_J) + \dim(\mathcal{C}^\perp|_J) - |J|. \tag{8}$$

Indeed, from Lemma 2 of [5], we have $\dim(\mathcal{C}|_{J^c}) = |J^c| -$

$\dim(\mathcal{C}^{\perp} \setminus J)$, and hence, by (5),

$$\begin{aligned}\lambda_{\mathcal{C}}(J) &= \dim(\mathcal{C}|_J) + |J^c| - \dim(\mathcal{C}^{\perp} \setminus J) - \dim(\mathcal{C}) \\ &= \dim(\mathcal{C}|_J) + \dim(\mathcal{C}^{\perp}) - \dim(\mathcal{C}^{\perp} \setminus J) - |J| \\ &= \dim(\mathcal{C}|_J) + \dim(\mathcal{C}^{\perp}|_J) - |J|,\end{aligned}$$

with the last equality following from Lemma 1 of [5].

We first prove the lemma in the case when $\mathcal{D} = \mathcal{C} \setminus X$ for some $X \subset E(\mathcal{C})$. From (8), we have for any $J \subset E(\mathcal{C} \setminus X)$,

$$\begin{aligned}\lambda_{\mathcal{C}\setminus X}(J) &= \dim((\mathcal{C} \setminus X)|_J) + \dim((\mathcal{C}^{\perp}/X)|_J) - |J| \\ &= \dim(\mathcal{C}|_{J\cup X}) - \dim(\mathcal{C}|_X) + \dim(\mathcal{C}^{\perp}|_J) - |J|,\end{aligned}$$

the second equality above being due to (2) and (3). Therefore, we have

$$\lambda_{\mathcal{C}}(J) - \lambda_{\mathcal{C}\setminus X}(J) = \dim(\mathcal{C}|_J) + \dim(\mathcal{C}|_X) - \dim(\mathcal{C}|_{J\cup X}). \tag{9}$$

Since the right-hand side above is always non-negative, we have that $\lambda_{\mathcal{C}}(J) \geq \lambda_{\mathcal{C}\setminus X}(J)$.

We next consider the case when $\mathcal{D} = \mathcal{C}/Y$ for some $Y \subset E(\mathcal{C})$. Since $(\mathcal{C}/Y)^{\perp} = \mathcal{C}^{\perp} \setminus Y$, we have $\lambda_{\mathcal{C}/Y} = \lambda_{\mathcal{C}^{\perp}\setminus Y}$. Therefore, for any $J \subset E(\mathcal{C}/Y)$,

$$\lambda_{\mathcal{C}}(J) - \lambda_{\mathcal{C}/Y}(J) = \lambda_{\mathcal{C}^{\perp}}(J) - \lambda_{\mathcal{C}^{\perp}\setminus Y}(J).$$

Hence, from (9) above, it follows that $\lambda_{\mathcal{C}}(J) - \lambda_{\mathcal{C}/Y}(J) = \dim(\mathcal{C}^{\perp}|_J) + \dim(\mathcal{C}^{\perp}|_Y) - \dim(\mathcal{C}^{\perp}|_{J\cup Y}) \geq 0$. Thus, we also have that $\lambda_{\mathcal{C}}(J) \geq \lambda_{\mathcal{C}/Y}(J)$.

Finally, consider any minor $\mathcal{D} = \mathcal{C} \setminus X/Y$. For any $J \subset E(\mathcal{D}) = E(\mathcal{C}) - (X \cup Y)$, we have

$$\lambda_{\mathcal{C}}(J) \geq \lambda_{\mathcal{C}\setminus X}(J) \geq \lambda_{(\mathcal{C}\setminus X)/Y}(J) = \lambda_{\mathcal{D}}(J),$$

as desired. ■

We can now prove the main result of this section.

Theorem III.2 *$TC_{w;q}$ is minor-closed.*

Proof: Let $\mathcal{C}$ be an arbitrary code in $TC_{w;q}$, so that $\sigma[\mathcal{C}] \leq w$, and let $\mathcal{D}$ be a minor of $\mathcal{C}$. We want to show that $[\mathcal{D}] \subset TC_{w;q}$.

Let $\mathcal{C}'$ be the code in $[\mathcal{C}]$ for which $s_{\max}(\mathcal{C}') = \sigma[\mathcal{C}]$. Clearly, $\mathcal{C}'$ has a minor $\mathcal{D}'$ that is in $[\mathcal{D}]$. It is enough to show that $\mathcal{D}' \in TC_{w;q}$, for it then follows from the fact that $TC_{w;q}$ is closed under code equivalence, that $[\mathcal{D}] = [\mathcal{D}'] \subset TC_{w;q}$.

Thus, without loss of generality, we may assume that $\mathcal{C}' = \mathcal{C}$ (so that $s_{\max}(\mathcal{C}) = \sigma[\mathcal{C}]$) and $\mathcal{D}' = \mathcal{D}$. Let $\alpha_{\mathcal{C}} = (\alpha_1, \ldots, \alpha_n)$.

Suppose first that $\mathcal{D} = \mathcal{C} \setminus \{\alpha_j\}$, for some $j \in [n]$. Then, $\alpha_{\mathcal{D}} = (\alpha_1, \ldots, \alpha_{j-1}, a_{j+1}, \ldots, \alpha_n)$. Now, for $1 \leq i < j$, we have from (6) and Lemma III.1,

$$s_i(\mathcal{D}) = \lambda_{\mathcal{D}}(\{\alpha_1, \ldots, \alpha_i\}) \leq \lambda_{\mathcal{C}}(\{\alpha_1, \ldots, \alpha_i\}) = s_i(\mathcal{C}).$$

And for $j \leq i \leq n-1$, using the same reasoning as above, as well as the fact that $\lambda_{\mathcal{D}}(J) = \lambda_{\mathcal{D}}(E(\mathcal{D}) - J)$, we have

$$\begin{aligned}s_i(\mathcal{D}) &= \lambda_{\mathcal{D}}(\{\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{i+1}\}) \\ &= \lambda_{\mathcal{D}}(\{\alpha_{i+2}, \ldots, \alpha_n\}) \\ &\leq \lambda_{\mathcal{C}}(\{\alpha_{i+2}, \ldots, \alpha_n\}) \\ &= \lambda_{\mathcal{C}}(\{\alpha_1, \ldots, \alpha_{i+1}\}) \quad = \quad s_{i+1}(\mathcal{C}).\end{aligned}$$

It follows that $\sigma[\mathcal{D}] \leq s_{\max}(\mathcal{D}) \leq s_{\max}(\mathcal{C}) = \sigma[\mathcal{C}] \leq w$, and hence, $\mathcal{D} \in TC_{w;q}$.

If $\mathcal{D} = \mathcal{C}/\{\alpha_j\}$ for some $j \in [n]$, the same argument as above shows that $\mathcal{D} \in TC_{w;q}$ as well. Thus, any minor obtained from $\mathcal{C}$ by puncturing or shortening at a single coordinate is in $TC_{w;q}$. It follows by a straightforward induction argument that if $\mathcal{D}$ is any minor of $\mathcal{C}$, then $\mathcal{D} \in TC_{w;q}$. ■

Recall that a code $\mathcal{D}$ is called an excluded minor (or sometimes, a forbidden minor) of the minor-closed family $TC_{w;q}$ if $\mathcal{D} \notin TC_{w;q}$, but every proper minor of $\mathcal{D}$ is in $TC_{w;q}$. It is fairly easy to see that a linear code $\mathcal{C}$ over $\mathbb{F}_q$ is in $TC_{w;q}$ iff it contains no minor that is an excluded minor of $TC_{w;q}$. The results of the next section show that the list of excluded minors for $TC_{w;q}$ is in fact finite. Thus, deciding membership of a code $\mathcal{C}$ in $TC_{w;q}$ is accomplished by testing whether or not $\mathcal{C}$ contains as a minor one of the finitely many excluded minors of $TC_{w;q}$.

IV. Trellis Complexity and Branchwidth

A notion closely related to state-complexity that has received considerable recent attention in the matroid theory literature, is that of branchwidth. We define branchwidth in the context of codes below, and provide an important application to state-complexity of matroid-theoretic results on branchwidth.

A *cubic tree* is a tree in which the degree of any vertex is either one or three. One of the degree-one vertices is distinguished as the *root* of the tree, while the remaining degree-one vertices are called *leaves*. The vertices of degree three are called *internal nodes*. For $n \geq 2$, let $\mathfrak{T}_n$, denote the set of all cubic trees with n leaves. For any cubic tree T, we shall let $\mathcal{E}(T)$ denote the set of its edges.

Let $\mathcal{C}$ be a linear code of length n over $\mathbb{F}_q$, with label set $E(\mathcal{C})$. Given a $T \in \mathfrak{T}_n$, let $\mathcal{L}$ be a one-to-one function, called a *labelling*, from the set of its leaves to $E(\mathcal{C})$. Note that the root and the internal nodes do not receive labels from $E(\mathcal{C})$. Each edge $e \in \mathcal{E}(T)$ connects two subtrees of T, so $T - e$ has two components. We say that edge e *displays* a subset $X \subset E(\mathcal{C})$ if X is the set of labels of leaves of one of the components of $T - e$. Note that if e displays X, then it also displays X^c. If X is displayed by e, then define $w_{\mathcal{C}}(e)$ to be $\lambda_{\mathcal{C}}(X)$, $\lambda_{\mathcal{C}}$ being the connectivity function of $\mathcal{C}$. Now, define the *width* of T, with respect to the labelling $\mathcal{L}$, to be

$$w_{\mathcal{C}}(T, \mathcal{L}) = \max_{e\in\mathcal{E}(T)} w_{\mathcal{C}}(e).$$

The *branchwidth* of $\mathcal{C}$ is defined to be the quantity

$$\beta[\mathcal{C}] = \min_{(T,\mathcal{L})} w_{\mathcal{C}}(T, \mathcal{L}) = \min_{(T,\mathcal{L})} \max_{e\in\mathcal{E}(T)} w_{\mathcal{C}}(e), \tag{10}$$

the minimum being taken over all pairs $(T, \mathcal{L})$ with $T \in \mathfrak{T}_n$ and $\mathcal{L}$ a labelling of T.

It is easily verified that if $\mathcal{C}'$ is a code equivalent to $\mathcal{C}$, then $\beta[\mathcal{C}'] = \beta[\mathcal{C}]$. This is because, given any $T \in \mathfrak{T}_n$, for each labelling $\mathcal{L}$ of T, there exists a labelling $\mathcal{L}'$ such that $w_{\mathcal{C}}(T, \mathcal{L}) = w_{\mathcal{C}'}(T, \mathcal{L}')$. So, branchwidth is really a characteristic of the equivalence class $[\mathcal{C}]$.

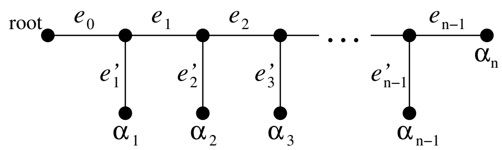

Fig. 1. The tree $\widehat{T}$, and the labelling $\widehat{\mathcal{L}}$.

Now, let $\mathcal{C}$ be a code with label sequence $\alpha_{\mathcal{C}} = (\alpha_1, \ldots, \alpha_n)$. Let $\widehat{T} \in \mathfrak{T}_n$ be the tree, and $\widehat{\mathcal{L}}$ its labelling, shown in Figure 1. It is clear that for $i \in \{0\} \cup [n-1]$, we have $w_{\mathcal{C}}(e_i) = \lambda_{\mathcal{C}}(\{\alpha_1, \ldots, \alpha_i\}) = s_i(\mathcal{C})$, while for $i \in [n-1]$, $w_{\mathcal{C}}(e'_i) = \lambda_{\mathcal{C}}(\{\alpha_i\})$. In particular, $w_{\mathcal{C}}(e'_i) \in \{0, 1\}$ for each $i \in [n-1]$.

Lemma IV.1 *If $w_{\mathcal{C}}(e'_i) = 1$ for some $i \in [n-1]$, then either $s_{i-1} \geq 1$ or $s_i \geq 1$.*

Proof: Using the fact that $\lambda_{\mathcal{C}}(J) = \lambda_{\mathcal{C}}(J^c)$, we have

$$\begin{aligned} s_{i-1} + s_i &= \lambda_{\mathcal{C}}(\{\alpha_i, \alpha_{i+1}, \ldots, \alpha_n\}) + \lambda_{\mathcal{C}}(\{\alpha_1, \ldots, \alpha_i\}) \\ &\geq \lambda_{\mathcal{C}}(\{\alpha_i\}) + \lambda_{\mathcal{C}}(E(\mathcal{C})) \quad = \quad w(e'_i), \end{aligned}$$

the inequality above arising from (7). The lemma directly follows. ∎

From the last lemma, it is evident that $w_{\mathcal{C}}(\widehat{T}, \widehat{\mathcal{L}}) = \max_{i \in [n-1]} s_i(\mathcal{C}) = s_{\max}(\mathcal{C})$. The following result is thus a direct consequence of the relevant definitions.

Lemma IV.2 *For any code $\mathcal{C}$, we have $\beta[\mathcal{C}] \leq \sigma[\mathcal{C}]$.*

In particular, any code in $TC_{w;q}$ has branchwidth at most w. Now, a deep matroid-theoretic result of Geelen and Whittle [7], when translated into coding-theoretic language, states the following.

Theorem IV.3 ([7], Theorem 1.4) *Let $\mathfrak{F}$ be a minor-closed family of linear codes over $\mathbb{F}_q$, and let k be a positive integer. If each code in $\mathfrak{F}$ has branchwidth at most k, then $\mathfrak{F}$ has finitely many excluded minors.*

In view of Theorem III.2, Lemma IV.2 and Theorem IV.3, the following result is now obvious.

Corollary IV.4 *For any finite field $\mathbb{F}_q$ and integer w, $TC_{w;q}$ has finitely many excluded minors.*

Therefore, as explained at the end of Section III, membership of a given code $\mathcal{C}$ in $TC_{w;q}$ can be decided by testing whether or not $\mathcal{C}$ contains as a minor one of finitely many codes. It is conjectured [6, Conjecture 1.3], [11, Conjecture 7.2] that the problem of deciding whether or not a given code $\mathcal{C}$ contains a fixed code $\mathcal{D}$ as a minor, can be solved in time polynomial in the length of $\mathcal{C}$. There is strong evidence in the literature in favour of the validity of this conjecture [6]. Clearly, if the conjecture is true, then the WTSC problem stated in Section I can be solved in polynomial time.

V. Excluded Minors for $TC_{1;2}$

In this section and the next, we restrict our attention to *binary* linear codes $\mathcal{C}$ with $\sigma[\mathcal{C}] \leq w$. To keep our notation simple, we will use TC_w, instead of $TC_{w;2}$, to denote the family of such codes. We will illustrate the practical difficulties involved in precisely determining the excluded minors of TC_w by considering the cases of $w = 1$ and $w = 2$ only. In this section, we deal with the family TC_1; the family TC_2 is considered in Section VI.

With a little effort, we can determine the complete list of excluded minors for TC_1. In order to do so, we need to bring in the concept of the cycle code of a graph. Let $\mathcal{G}$ be an undirected graph with vertex set $V(\mathcal{G}) = \{v_1, \ldots, v_m\}$ and edge set $E(\mathcal{G}) = \{e_1, \ldots, e_n\}$. Let $A_{\mathcal{G}}$ be the vertex-edge incidence matrix of $\mathcal{G}$, *i.e.*, the $m \times n$ matrix whose (i,j)th entry, $a_{i,j}$, is 1 if vertex v_i is incident with edge e_j; and $a_{i,j} = 0$ otherwise. The *cycle code*, $\mathcal{C}(\mathcal{G})$, of $\mathcal{G}$ is the binary linear code that has $A_{\mathcal{G}}$ as a parity-check matrix. The reason for the nomenclature is that a binary word $(c_1, \ldots, c_n)$ is in $\mathcal{C}(\mathcal{G})$ iff the set of edges $\{e_j : c_j = 1\}$ forms a cycle in $\mathcal{G}$. The label sequence of $\mathcal{C}(\mathcal{G})$ is taken to be $\alpha_{\mathcal{C}(\mathcal{G})} = (e_1, \ldots, e_n)$, and hence, $E(\mathcal{C}(\mathcal{G})) = E(\mathcal{G})$. A code $\mathcal{C}$ is said to be *graphic* if it is the cycle code of some graph.

We record here some useful facts about cycle codes. Proofs of these can be found in [16], albeit couched in the language of matroid theory. If a code $\mathcal{C}$ is graphic, then there is a *connected* graph $\mathcal{G}$ such that $\mathcal{C} = \mathcal{C}(\mathcal{G})$. When $\mathcal{G}$ is connected, the rank of the matrix $A_{\mathcal{G}}$ is $|V(\mathcal{G})| - 1$, and therefore, $\dim(\mathcal{C}(\mathcal{G})) = |E(\mathcal{G})| - |V(\mathcal{G})| + 1$. Furthermore, for any $J \subset E(\mathcal{G})$, if $A_{\mathcal{G}}|_J$ denotes the matrix obtained by restricting $A_{\mathcal{G}}$ to the columns labelled by the edges in J, then

$$\mathsf{rank}(A_{\mathcal{G}}|_J) = \dim(\mathcal{C}(\mathcal{G})^{\perp}|_J) = r(J), \tag{11}$$

where $r(J)$ denotes the number of edges in any spanning forest of the subgraph of $\mathcal{G}$ induced by J. To be precise, letting $\mathcal{G}[J]$ denote the subgraph of $\mathcal{G}$ induced by J, we have $r(J) = |V(\mathcal{G}[J])| - \omega(\mathcal{G}[J])$, where $\omega(\mathcal{G}[J])$ is the number of connected components of $\mathcal{G}[J]$. The following useful lemma is now immediately obvious from the definition of $\lambda_{\mathcal{C}^{\perp}}$.

Lemma V.1 *Let w be a positive integer, and let $\mathcal{C} = \mathcal{C}(\mathcal{G})$ for a connected graph $\mathcal{G}$. Then, $\mathcal{C}^{\perp}$ (and hence, $\mathcal{C}$) is in TC_w iff there is an ordering $(e_1, e_2, \ldots, e_n)$ of the edges of $\mathcal{G}$ such that $\forall j \in [n]$, $r(e_1, \ldots, e_j) + r(e_{j+1}, \ldots, e_n) \leq |V(\mathcal{G})| + w - 1$.*

Given $e \in E(\mathcal{G})$, define the graph $\mathcal{G} \setminus e$ to be the graph obtained by deleting the edge e along with any vertices that get isolated as a result of deleting e. Also, define $\mathcal{G}/e$ to be the graph obtained by contracting e, *i.e.*, deleting e and identifying the two vertices incident with e. A minor of the graph $\mathcal{G}$ is any graph obtained from $\mathcal{G}$ via a (possibly empty) sequence of edge deletions and contractions. The operations of edge deletion and contraction are the graphic analogues of code shortening and puncturing, respectively. To be precise, for any $e \in E(\mathcal{G})$, we have

$$\mathcal{C}(\mathcal{G})/e = \mathcal{C}(\mathcal{G}/e) \quad \text{and} \quad \mathcal{C}(\mathcal{G}) \setminus e = \mathcal{C}(\mathcal{G} \setminus e). \tag{12}$$

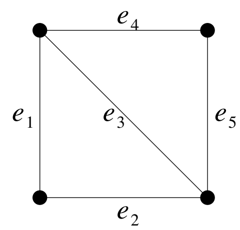

Fig. 2. The graph $K_4 \setminus e$.

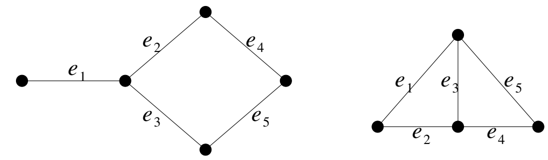

Fig. 3. The graphs $K_{2,3} \setminus e$ and $K_{2,3}/e$.

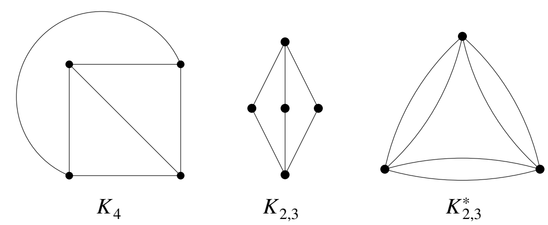

Fig. 4. The graphs whose cycle codes are excluded minors for TC_1.

It follows that any minor of a graphic code is graphic.

We are now in a position to state the main result of this section, which is an excluded-minor characterization of TC_1. As is usual, we let K_n denote the complete graph on n vertices, and $K_{m,n}$ the complete bipartite graph with m vertices in one part, and n in the other.

Theorem V.2 *A binary linear code is in TC_1 iff it contains no minor equivalent to $\mathcal{C}(K_4)$, $\mathcal{C}(K_{2,3})$ or $\mathcal{C}(K_{2,3})^\perp$.*

We first prove the easier "only if" part of the above theorem. This is accomplished by the following proposition.

Proposition V.3 *The codes $\mathcal{C}(K_4)$, $\mathcal{C}(K_{2,3})$ and $\mathcal{C}(K_{2,3})^\perp$ are excluded minors for TC_1.*

Proof: Let $(e_1, e_2, \ldots, e_6)$ be an ordering of the edges of K_4. Then, $r(e_1, e_2, e_3) \geq 2$, with equality iff $\{e_1, e_2, e_3\}$ forms a triangle, in which case $\{e_4, e_5, e_6\}$ forms a triad (*i.e.* a tree of three edges all incident with a common vertex). It follows that $r(e_1, e_2, e_3) + r(e_4, e_5, e_6) \geq 5$. Hence, by Lemma V.1, $\mathcal{C}(K_4)$ is not in TC_1.

To show that any proper minor of $\mathcal{C}(K_4)$ is in TC_1, it is enough to show that for any $e \in E(K_4)$, $\mathcal{C}(K_4)/e$ and $\mathcal{C}(K_4) \setminus e$ are in TC_1. Now, $\mathcal{C}(K_4)$ is the binary linear code generated by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}. \tag{13}$$

It is obvious that $\mathcal{C}(K_4)$ is equivalent to its dual, and hence, $(\mathcal{C}(K_4)/e)^\perp = \mathcal{C}(K_4)^\perp \setminus e \equiv \mathcal{C}(K_4) \setminus e$. Therefore, we only need to show that $\mathcal{C}(K_4) \setminus e = \mathcal{C}(K_4 \setminus e)$ is in TC_1. For any $e \in E(K_4)$, $K_4 \setminus e$ is isomorphic to the graph shown in Figure 2, and the ordering $(e_1, \ldots, e_5)$ of the edges shown in the figure satisfies the condition of Lemma V.1. Thus, $\mathcal{C}(K_4) \setminus e$ is in TC_1, and hence, $\mathcal{C}(K_4)$ is an excluded minor for TC_1.

The proof for $\mathcal{C}(K_{2,3})$ and $\mathcal{C}(K_{2,3})^\perp$ is very similar. For any $J \subset E(K_{2,3})$ with $|J| = 3$, it is easy to verify that $r(J) = 3$. Therefore, for any partition (J, J^c) of $E(K_{2,3})$, with $|J| = |J^c| = 3$, we must have $r(J) + r(J^c) = 6$. Hence, by Lemma V.1, neither $\mathcal{C}(K_{2,3})$ nor $\mathcal{C}(K_{2,3})^\perp$ is in TC_1.

For any $e \in E(K_{2,3})$, the graphs $K_{2,3}/e$ and $K_{2,3} \setminus e$ are as shown in Figure 3. The ordering $(e_1, \ldots, e_5)$ of the edges shown in the figure satisfies the condition of Lemma V.1, and hence, $\mathcal{C}(K_{2,3})/e$ and $\mathcal{C}(K_{2,3}) \setminus e$ are both in TC_1. It follows that $\mathcal{C}(K_{2,3})$ is an excluded minor for TC_1, and therefore, so is $\mathcal{C}(K_{2,3})^\perp$. ∎

To prove the converse part of Theorem V.2, we need to show that the codes listed in Proposition V.3 constitute all the excluded minors of TC_1. For the remainder of the proof of Theorem V.2, we take $\mathcal{C}$ to be a binary linear code that contains no minor equivalent to $\mathcal{C}(K_4)$, $\mathcal{C}(K_{2,3})$ or $\mathcal{C}(K_{2,3})^\perp$. Note that $\mathcal{C}^\perp$ also cannot contain a minor equivalent to any of $\mathcal{C}(K_4)$, $\mathcal{C}(K_{2,3})$ and $\mathcal{C}(K_{2,3})^\perp$. Our goal is to show that $\mathcal{C} \in TC_1$, or equivalently, that $\mathcal{C}^\perp \in TC_1$.

It is easily verified that $\mathcal{C}(K_4)$ is a minor of the [7,4] Hamming code $\mathcal{H}_7$: shortening $\mathcal{H}_7$ at any coordinate yields a code equivalent to that generated by the matrix in (13). Since K_4 (as a graph) is a minor of K_5 as well as $K_{3,3}$, we find that, by (12), $\mathcal{C}(K_4)$ is a minor of the codes $\mathcal{C}(K_5)$ and $\mathcal{C}(K_{3,3})$. Furthermore, since $\mathcal{C}(K_4)$ is equivalent to its dual, it is also a minor of the codes $\mathcal{H}_7^\perp$, $\mathcal{C}(K_5)^\perp$ and $\mathcal{C}(K_{3,3})^\perp$. Hence, $\mathcal{C}$ contains no minor equivalent to any of the codes $\mathcal{H}_7$, $\mathcal{H}_7^\perp$, $\mathcal{C}(K_5)$, $\mathcal{C}(K_5)^\perp$, $\mathcal{C}(K_{3,3})$ and $\mathcal{C}(K_{3,3})^\perp$. Therefore, by Theorem 13.3.3 and Proposition 5.2.6 in [16], $\mathcal{C} = \mathcal{C}(\mathcal{G})$ for some planar graph $\mathcal{G}$. Evidently, we may take $\mathcal{G}$ to be connected as a graph.

Since TC_1 is closed under direct sums, we may assume that $\mathcal{C}$ is *connected* as a code, *i.e.*, it cannot be expressed as the direct sum of smaller codes. Therefore, $\mathcal{G}$ is either a graph consisting of a single vertex with a self-loop incident with it, or $\mathcal{G}$ is a loopless graph. Indeed, if $|V(\mathcal{G})| \geq 2$, and e were a self-loop in $\mathcal{G}$, then $\mathcal{C} \equiv \mathcal{C}(\mathcal{G} \setminus e) \oplus \{0, 1\}$. The cycle code of a graph consisting of a single self-loop is just $\{0, 1\}$, which is obviously in TC_1. So, we may assume that $\mathcal{G}$ is loopless. It is a simple matter to check, using Lemma V.1, that for all loopless, connected graphs $\mathcal{G}$ on two vertices, $\mathcal{C}(\mathcal{G}) \in TC_1$. Hence, we may assume that $|V(\mathcal{G})| \geq 3$, in which case, by Corollary 8.2.2[2] in [16], $\mathcal{G}$ is 2-connected as a graph.

At this point, we need the following definition. A graph is called an *umbrella* if it is of the form shown in Figure 5.

[2]Corollary 8.2.2 in [16] states that for a graph $\mathcal{G}$ with $|V(\mathcal{G})| \geq 3$, and no isolated vertices, $\mathcal{C}(\mathcal{G})$ is connected as a code iff $\mathcal{G}$ is 2-connected as a graph.

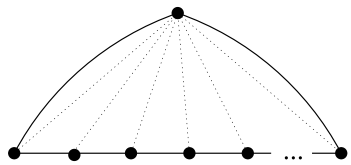

Fig. 5. An "umbrella" graph. A dotted line between a pair of vertices represents zero or more parallel edges between them.

More precisely, an umbrella is a graph H that consists of a circuit on $m+1$ vertices $u_0, u_1, \ldots, u_m$, and in addition, for each $i \in [m]$, zero or more parallel edges between u_0 and u_i. Note that $H - u_0$ is a simple path, where $H - u_0$ denotes the graph obtained from H by deleting the vertex u_0 and all edges incident with it.

The role of umbrellas in our proof is evident from the next lemma concerning the loopless, 2-connected, planar graph $\mathcal{G}$ such that $\mathcal{C} = \mathcal{C}(\mathcal{G})$. The lemma also requires the well-known graph-theoretic notion of the geometric dual of a planar graph (see *e.g.* [8, p. 113] or [16, p. 91]).

Lemma V.4 *$\mathcal{G}$ has a geometric dual $\mathcal{G}^*$ that is isomorphic to an umbrella.*

We prove the lemma using the concept of an outerplanar graph. A planar graph is said to be *outerplanar* if it has a planar embedding in which every vertex lies on the exterior (unbounded) face. We will refer to such a planar embedding of the graph as an *outerplanar embedding*. Outerplanar graphs were characterized by Chartrand and Harary [2] as graphs that do not contain K_4 or $K_{2,3}$ as a minor.

Proof of Lemma V.4: Since $\mathcal{C}$ contains no minor equivalent to $\mathcal{C}(K_4)$ or $\mathcal{C}(K_{2,3})$, by (12), $\mathcal{G}$ cannot contain K_4 or $K_{2,3}$ as a minor. Therefore, by the Chartrand-Harary result mentioned above, $\mathcal{G}$ is outerplanar. Let $\mathcal{G}^*$ be the geometric dual of an outerplanar embedding of $\mathcal{G}$.

Now, $\mathcal{C}(\mathcal{G}^*) \equiv \mathcal{C}^\perp$. Since $\mathcal{C}$ is connected, so is $\mathcal{C}^\perp$, and hence $\mathcal{G}^*$ is loopless as well. If $|V(G^*)| = 2$, then there is nothing to prove, so we may assume that $|V(G^*)| \geq 3$. Hence, by Corollary 8.2.2 in [16], $\mathcal{G}^*$ is 2-connected as a graph.

Let x be the vertex of $\mathcal{G}^*$ corresponding to the exterior face of the outerplanar embedding of $\mathcal{G}$. By a result of Fleischner *et al.* [3, Theorem 1], $\mathcal{G}^* - x$ is a forest. In fact, since $\mathcal{G}^*$ is 2-connected, $\mathcal{G}^* - x$ is a tree.

We claim that no vertex of $\mathcal{G}^* - x$ has degree greater than two, and hence, $\mathcal{G}^* - x$ is a simple path. Indeed, suppose that $\mathcal{G}^* - x$ has a vertex u adjacent to three other vertices v_1, v_2, v_3. Since G^* is 2-connected, there are paths π_1, π_2 and π_3 in $\mathcal{G}^*$ from v_1, v_2 and v_3, respectively, to x that do not pass through u. Also, since $\mathcal{G}^* - x$ is a tree, these paths must be internally disjoint in $\mathcal{G}^*$. The graph $\mathcal{G}^*$ thus has a subgraph as depicted in Figure 6. But this subgraph is obviously contractible to $K_{2,3}$, and hence $\mathcal{G}^*$ has $K_{2,3}$ as a minor. However, this is impossible, as $\mathcal{C}(\mathcal{G}^*) \equiv \mathcal{C}^\perp$, and $\mathcal{C}^\perp$ does not have $\mathcal{C}(K_{2,3})$ as a minor.

Thus, $\mathcal{G}^* - x$ is a simple path. The two degree-one vertices (end-points) of this path must be adjacent to x in $\mathcal{G}^*$; otherwise, $\mathcal{G}^*$ is not 2-connected. It follows that $\mathcal{G}^*$ is isomorphic to an umbrella. ■

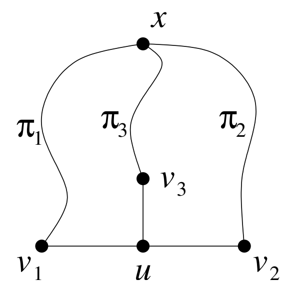

Fig. 6. If $\mathcal{G}^* - x$ has a vertex of degree at least 3, then $\mathcal{G}^*$ has a $K_{2,3}$ minor.

Now, to complete the proof of Theorem V.2, it is enough to show that $\mathcal{C}(\mathcal{G}^*) \in TC_1$, since $\mathcal{C}^\perp \equiv \mathcal{C}(\mathcal{G}^*)$. This is done by the following lemma.

Lemma V.5 *If H is an umbrella, then $\mathcal{C}(H) \in TC_1$.*

Proof: Let H be an umbrella on $m+1$ vertices $u_0, u_1, \ldots, u_m$, where u_0 is the vertex such that $H - \{u_0\}$ is a simple path. For $i \in [m]$, let E_i denote the set of edges between u_0 and u_i. Also, for $j \in [m-1]$, let e_j denote the edge between u_j and u_{j+1}. Consider the ordering of the edges of H given by

$$(E_1, e_1, E_2, e_2, \ldots, E_{m-1}, e_{m-1}, E_m),$$

where it is understood that for each $i \in [m]$, the edges within E_i (if any) are given an arbitrary order.

We will apply Lemma V.1. Consider any $J = \left(\bigcup_{i=1}^{j-1} (E_i \cup \{e_i\})\right) \cup X$, with $X \subset E_j$ (X may be empty). We want to determine $r(J) + r(E(H) - J)$. Note that the subgraph, $H[J]$, of H induced by the edges in J is incident only with vertices in $\{u_0, u_1, \ldots, u_j\}$. Therefore, $r(J) = |V(H[J])| - 1 \leq j$. Similarly, the subgraph of H induced by the edges in $E(H) - J$ is incident only with vertices in $\{u_j, u_{j+1}, \ldots, u_m, u_0\}$, and so, $r(E(H) - J) \leq m - j + 1$. Therefore, $r(J) + r(E(H) - J) \leq m + 1 = |V(H)|$. It follows from Lemma V.1 that $\mathcal{C}(H) \in TC_1$. ■

The proof of Theorem V.2 is now complete.

The characterization given in Theorem V.2 can be used to derive a polynomial-time algorithm that, given a generator (or parity-check matrix) for a binary code $\mathcal{C}$, determines whether or not $\mathcal{C}$ is in TC_1. This is based on the following lemma, which can be deduced from Theorem V.2 and the aforementioned excluded-minor characterization of outerplanar graphs due to Chartrand and Harary [2].

Lemma V.6 *Let $\mathcal{C}$ be a binary linear code such that $\mathcal{C} = \mathcal{C}(\mathcal{G})$ and $\mathcal{C}^\perp = \mathcal{C}(\widetilde{\mathcal{G}})$ for some graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$. Then, $\mathcal{C} \in TC_1$ (equivalently, $\mathcal{C}^\perp \in TC_1$) iff $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are outerplanar.*

The algorithm outlined below takes as input a $k \times n$ generator matrix for a binary linear code $\mathcal{C}$, and decides whether or not $\mathcal{C}$ is in TC_1.

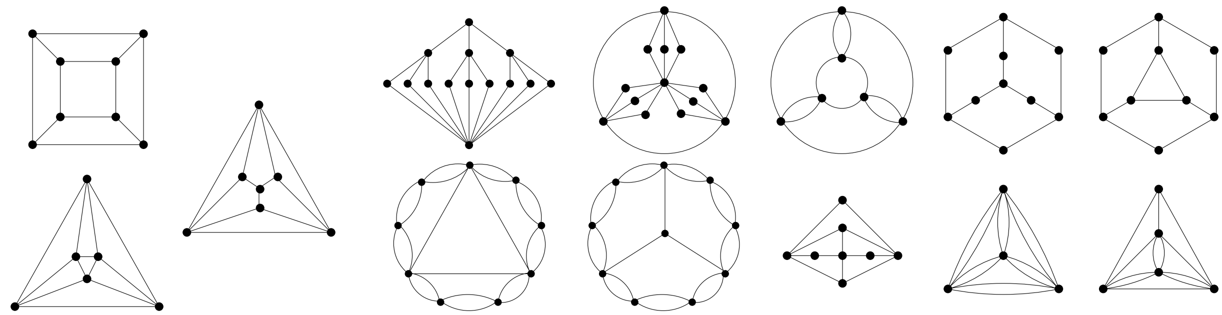

Fig. 7. Some of the planar graphs whose cycle codes are excluded minors for TC_2.

Step 1: Determine whether $\mathcal{C}$ and $\mathcal{C}^\perp$ are graphic; if either of them is not, then $\mathcal{C} \notin TC_1$, STOP; if both are graphic, determine graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ such that $\mathcal{C} = \mathcal{C}(\mathcal{G})$ and $\mathcal{C}^\perp = \mathcal{C}(\widetilde{\mathcal{G}})$.

Step 2: Determine whether $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are outerplanar; if either of them is not, then $\mathcal{C} \notin TC_1$, STOP; if both are outerplanar, then $\mathcal{C} \in TC_1$.

There are efficient procedures known [1], [14] for performing both steps of the algorithm. From the running times of these procedures, we determine that the above algorithm can be implemented to run in $O(k^2(n-k)^2n^2)$ time. We remark that the above algorithm may be easily extended to determine the code $\mathcal{C}'$ equivalent to $\mathcal{C}$ such that $s_{\max}(\mathcal{C}') \le 1$.

VI. Excluded Minors for $TC_{2;2}$

While we are able to give complete excluded-minor and algorithmic characterizations for codes in TC_1, the same is not yet the case for TC_2. The methods of the previous section can be used to show the following result.

Proposition VI.1 *The codes $\mathcal{H}_7$, $\mathcal{H}_7^\perp$, $\mathcal{C}(K_5)$, $\mathcal{C}(K_5)^\perp$, $\mathcal{C}(K_{3,3})$, $\mathcal{C}(K_{3,3})^\perp$, and $\mathcal{C}(\mathcal{G})$, where $\mathcal{G}$ is any of the planar graphs in Figure 7, are excluded minors for TC_2.*

However, the list of excluded minors above does not appear to be complete. The following corollary to the above proposition gives a necessary condition that must be satisfied by an excluded minor not listed in the proposition.

Corollary VI.2 *If $\mathcal{D}$ is an excluded minor of TC_2 not listed in Proposition VI.1, then $\mathcal{D} = \mathcal{C}(\mathcal{G})$ for some planar graph $\mathcal{G}$ that does not contain as a minor any of the graphs in Figure 7.*

The above corollary is a direct consequence of Theorem 13.3.3 and Proposition 5.2.6 in [16]. For the same reason, we also have the following necessary condition for a code to be in TC_2: $\mathcal{C} \in TC_2$ only if $\mathcal{C} = \mathcal{C}(\mathcal{G})$ for some planar graph $\mathcal{G}$ that does not contain as a minor any of the graphs in Figure 7.

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