Lecture 5: Functions : Images, Compositions, Inverses

1 Functions

We have all seen some form of functions in high school. For example, we have seen polynomial, exponential, logarithmic, trigonometric functions in calculus. These functions map real numbers to real numbers. We have also seen functions that map functions to functions, such a derivatives. Functions are of interest in many branches of mathematics, including enumerative combinatorics, topology, and group theory among others. An abstract understanding of function would be, an output $f(x)$ for each input x. We formalize this notion below.

Definition 1.1 (Function). Let A and B be sets. A **function** (also called a **map**) from A to B denoted $f : A \to B$ is a subset $F \subseteq A \times B$ such that for each $a \in A$, there is a unique pair of the form (a, b) in F. The set A is called the domain of f and the set B is called the co-domain of f.

Remark 1. To show the equality of functions, we need to show that the domain, co-domain, and subset of the product of domain and co-domain satisfying the given condition imposed by function f, all three must agree.

Definition 1.2. Let A and B be sets, and let $S \subseteq A$ be a subset.

1. A **constant map** $f : A \to B$ is any function of the form $f(x) = b$ for all $x \in A$, where $b \in B$ is some fixed element.

2. The **identity map** on A is the function $1_A : A \to A$ defined by $1_A(x) = x$ for all $x \in A$.

3. The **inclusion map** from $S \to A$ is the function $j : S \to A$ defined by $j(x) = x$ for all $x \in A$.

4. If $f : A \to B$ is a map, the **restriction** of $f \to S$, denoted by $f|_S$ is the map $f|_S : S \to B$, defined by $f|_S(x) = x$ for all $x \in S$.

5. If $g : A \to B$ is a map, an **extension** of g to A is any map $G : A \to B$ such that $G|_s = g$.

6. The **projection maps** from $A \times B$ are the functions, $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ defined by $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$ for all $(a, b) \in A \times B$. Projection maps $\pi_i : A_1 \times A_2 \times \cdots \times A_p \to A_i$ for any finite collection of sets $A_1, A_2 \dots A_p$ are defined similarly.

2 Image and Inverse Image

Definition 2.1 (Image and inverse image). Let $f : A \to B$ be a function.

1. For each $P \subseteq A$, the **image** of P under f is defined as

$$f(P) = \{b \in B : b = f(p) \text{ for some } p \in P\} = \{f(p) : p \in P\}.$$

2. For each $Q \subseteq B$, the **inverse image** (or **preimage**) of Q under f is defined as

$$f^{-1}(Q) = \{a \in A : f(a) = q \text{ for some } q \in Q\} = \{a \in A : f(a) \in Q\}$$

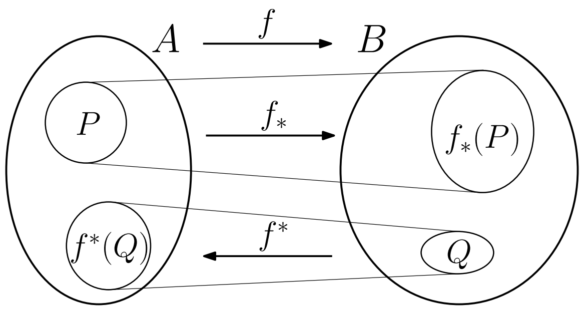

Figure 1: Image and inverse image under a function f.

Remark 2. Let $f : A \to B$ be a function.

i) For every $\emptyset \neq P \subseteq A$, $\emptyset \neq f(P) \subseteq B$ and $|P| \geqslant |f(P)|$.

ii) The **range** (or **image**) of f is the set $f(A)$. The range need not be equal to the co-domain.

iii) For every $Q \subseteq B$, $f^{-1}(Q) \subseteq A$, possibly be empty, and $|Q| \leqslant |f^{-1}(Q)|$.

iv) Given a function $f : A \to B$, the process of taking image of subsets of A can be thought of as operation of a function $f_* : \mathcal{P}(A) \to \mathcal{P}(B)$ on subsets of A and induced by f.

v) Given a function $f : A \to B$, the process of taking inverse image of subsets of B can be thought of as operation of a new function $f^* : \mathcal{P}(B) \to \mathcal{P}(A)$ on subsets of B and induced by f.

vi) Abuse of notation:

a) Notice that f maps elements in A to elements in B, and f_* maps subsets of A to subsets of B. Following common convetions, we will use f for $f_{/ast}$.

b) Similarly, f^* is replaced with f^{-1} for inverse image.

c) Later, we will look at the **inverse** of a function $f : A \to B$ (if it exists) and denote it by $f^{-1} : B \to A$. If the inverse function of f does not exist, then $f^{-1}(Q)$ is used to refer to the inverse image $f^{-1}(Q)$ of Q under f. If the inverse of f exists, then it takes elements and not subsets of B as the argument, and $f^{-1}(Q) = f^*(Q) = f_*^{-1}(Q)$. That is, $f^{-1}(Q)$ can be used to refer to both the inverse image $f^*(Q)$ of Q under f and the image $f_*^{-1}(Q)$ of Q under f^{-1}.

Example 2.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ plotted in Figure 2.

i) The range of f, $f(\mathbb{R}) = [-3, \infty) \subset \mathbb{R}$.

ii) For $P_1 = [1.5, 1.9]$ and $P_2 = [-4.5, -3.3]$, $f(P_1) = [1.7, 2.5]$ and $f(P_2) = [-3, -1]$.

iii) For $Q_1 = [1.7, 2.5]$ and $Q_2 = [-4, -3.2]$, $f^{-1}(Q_1) = [-2, -1.6] \cup [-0.6, 0] \cup [1.7, 2.5]$ and $f^{-1}(Q_2) = \emptyset$.

iv) From (i) and (ii), we have that $f^{-1}(f(P_1)) \neq P_1$, $f^{-1}(f(P_2)) = P_2$, $f(f^{-1}(Q_1)) = Q_1$ and $f(f^{-1}(Q_2)) = Q_2$ (cf. Theorem 2.3).

We state the following theorem with proof left as an exercise to the reader. Most of the proofs require showing set equality using set inclusions.

Theorem 2.3. *Let A and B be sets, let $C, D \subseteq A$ and $S, T \subseteq B$ be subsets of A and B respectively, and let $f : A \to B$ be a function. Let $I, J \neq \emptyset$, let $\{U_i : i \in I\}$ and $\{V_j : j \in J\}$ be indexed families of sets, where $U_i \subseteq A$, for all $i \in I$ and $V_j \subseteq B$, for all $j \in J$.*

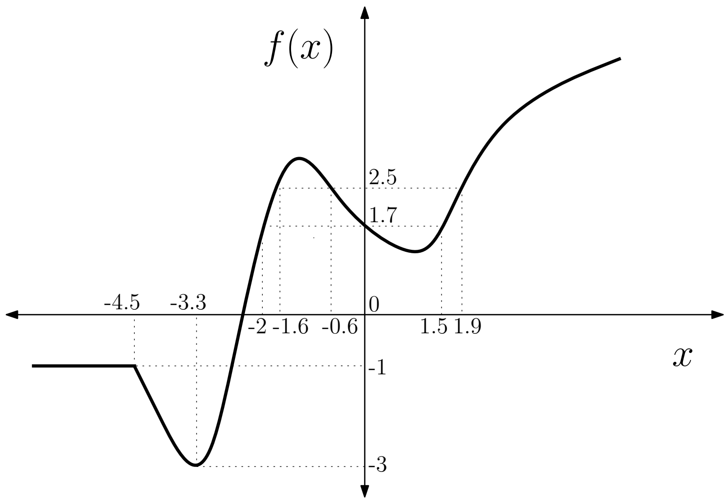

Figure 2: Example function.

i) $f(\emptyset) = \emptyset$ *and* $f^{-1}(\emptyset) = \emptyset$.

ii) $f^{-1}(B) = A$.

iii) $f(C) \subseteq S$ *iff* $C \subseteq f^{-1}(S)$.

iv) *If* $C \subseteq D$, *then* $f(C) \subseteq f(D)$.

v) *If* $S \subseteq T$, *then* $f^{-1}(S) \subseteq f^{-1}(T)$.

vi) $f(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f(U_i)$.

vii) $f(\bigcap_{i\in I} U_i) \subseteq \bigcap_{i\in I} f(U_i)$.

viii) $f^{-1}(\bigcup_{j\in J} V_j) = \bigcup_{j\in J} f^{-1}(V_j)$.

ix) $f^{-1}(\bigcap_{j\in J} V_j) = \bigcap_{j\in J} f^{-1}(V_j)$.

3 Composition of Function

We are interested in combining functions to create more interesting functions. Addition and multiplication of functions is not defined on arbitrary sets. However, as we will see in this section, combination is a natural way to define new functions on arbitrary sets.

Definition 3.1 (Composition of Functions). Let A, B and C be sets, and let $f : A \to B$ and $g : B \to C$ be functions. The **composition** of f and g is the function $g \circ f : A \to C$ defined as

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A.$$

Function compositions can be visualized by **commutative diagrams**. Figure 3 has the commutative diagram for $g \circ f$.

Remark 3. Let $f : A \to B$ and $g : B \to C$ be functions.

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

Figure 3: Commutative diagram for $g \circ f$.

i) Composition of three or more functions can be defined similarly.

ii) Though read/written left to write, while obtaining value of $(g \circ f)(a), a \in A$, $f(a)$ is computed first followed by $g(f(a))$.

iii) The composition $g \circ f$ is defined iff the range of f is a subset of the domain of g.

iv) If $A = C$, then $f \circ g$ is also defined but need not necessarily be equal to $g \circ f$ or 1_A.

v) The range of $g \circ f$ is a subset of range of g. That is, $(g \circ f)(A) \subseteq g(B)$.

Example 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$, $g : [0, \infty) \to \mathbb{R}$ be defined by $g(x) = \sqrt{x}$ and $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = 2x$. Then

i) $f \circ f : \mathbb{R} \to \mathbb{R}$ and $(f \circ f)(x) = (x^3)^3$.

ii) $(f \circ g) : [0, \infty) \to \mathbb{R}$ and $(f \circ g)(x) = \sqrt{x^3}$.

iii) $(f \circ h) : \mathbb{R} \to \mathbb{R}$ and $(f \circ h)(x) = (2x)^3$.

iv) $(h \circ f) : \mathbb{R} \to \mathbb{R}$ and $(h \circ f)(x) = 2x^3$. (Note that $(h \circ f) \neq (f \circ h)$)

v) $(h \circ g) : [0, \infty) \to \mathbb{R}$ and $(h \circ g)(x) = 2\sqrt{x}$.

vi) $f \circ h \circ g : [0, \infty) \to \mathbb{R}$ and $(f \circ h \circ g)(x) = (2\sqrt{x})^3$.

vii) $f \circ f \circ h : \mathbb{R} \to \mathbb{R}$ and $(f \circ f \circ h)(x) = ((2x)^3)^3$.

viii) $(g \circ g)$, $(g \circ f)$, $(g \circ h)$, $(f \circ g \circ h)$, $(g \circ h \circ f)$, $(g \circ f \circ h)$, $(h \circ g \circ f)$ are not defined.

Similarly, many more functions can be obtained.

Definition 3.3 (Coordinate Function). Let $A, A_1, A_2, \ldots, A_n$ be sets for some $n \in \mathbb{N}$, and let $f : A \to A_1 \times A_2 \times \ldots \times A_n$ be a function. For each $i \in \{1, 2, \ldots, n\}$, let $f_i : A \to A_i$ be defined by $f_i = \pi_i \circ f$, where $\pi_i : A_1 \times A_2 \times \ldots \times A_n \to A$ is the i^{th} projection map. Then, functions $f_1, f_2, \ldots, f_n$ are the **coordinate functions** of f.

With some abuse of notation, function f is sometimes written in terms of it's co-ordinate functions as $(f_1, f_2, \ldots, f_n)$ or $f_1 \times f_2 \times \ldots \times f_n$. Coordinates functions can be represented using a commutative diagram; given below is the commutative diagram for $n = 2$.

$$\begin{tikzcd}
& A \arrow[dl, "f_1"'] \arrow[d, "f"'] \arrow[dr, "f_2"] & \\
A_1 & A_1 \times A_2 \arrow[l, "\Pi_1"] \arrow[r, "\Pi_2"'] & A_2
\end{tikzcd}$$

Example 3.4. The function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (xy,\ sin(x^2),\ x+y^3)$ has 3 coordinate functions $f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f_1((x, y)) = xy, f_2((x, y)) = sin(x^2), \text{ and } f_3((x, y)) = x + y^3.$$

3.1 Properties of Composition of Functions

We wish to see whether properties such commutativity and associativity hold for this function operation. It turns out that associativity holds, but commutativity does not always hold for function composition (see Example 3.2(iv)).

Lemma 3.5. *Let A, B, C, D be sets and $f :\to B$, $g : B \to C$ and $h : C \to D$ be functions. Then the following are true.*

i) (Associative law) $(h \circ g) \circ f = h \circ (g \circ f)$.

ii) (Identity law) $f \circ 1_A = f$ and $1_B \circ f = f$.

4 Inverse Function

We are interested in finding out conditions for existence of inverse of a function f. We will also see that this inverse is unique when it exists.

Definition 4.1 (Existence). Let A and B be sets and let $f : A \to B$ and $g : B \to A$ be functions. Then the function g is

i) a **right inverse** for f if $f \circ g = 1_B$,

ii) a **left inverse** for f if $g \circ f = 1_A$, and

iii) an **inverse** for f if it is both a right and left inverse.

Remark 4. If g is a left (right) inverse for f, then f is a right (left) inverse for g.

Example 4.2. Let P be the set of all people and W be the set of women with at least one child, and let $c : P \to W$ be the function that maps a person to their mother and $m : W \to P$ be the function that maps a woman to her eldest child.

Choose a person $p \in P$ who is not the eldest of their siblings, and let the eldest sibling of the chosen person be p'. Then, $c(p) = w$, for some $w \in W$ and $m(w) = p' \neq p$. Thus, $(m \circ c)(p) \neq p, \forall p \in P$.

Choose a woman $w \in W$. Then, $m(w) = p$ for some $p \in P$ and $c(p) = w$. Thus, $(c \circ m)(w) = w$.

Hence, m has a left inverse (c) but no right inverse and c has a right inverse (m) but no left inverse.

Had P been the set of people who are eldest of their siblings, then the maps c and m would have been inverse of each other.

Had m been a map from P to P, then neither right nor left inverse of either maps would not have existed.

Lemma 4.3 (Uniqueness). *Let A and B be sets, and let $f : A \to B$ be a function.*

i) If f has an inverse, then it is unique.

ii) If f has a right inverse g and a left inverse h, then $g = h$, and hence f has an inverse.

iii) If f has an inverse g, then g has an inverse, which is f.

Proof. Same proof can be used in parts i) and ii). Part iii) is left as an exercise to the reader. Suppose $g, h : B \to A$ are both inverses of f. We will show $g = h$. By virtue of being inverses, g and h should be right and left inverse of f respectively. That is, $f \circ g = 1_B$ and $h \circ f = 1_A$. Using associativity of combination of functions, we conclude,

$$g = 1_A \circ g = h \circ f \circ g = h \circ 1_B = h.$$

□

Definition 4.4. Let A and B be sets, and let $f : A \to B$ be a function. If f has an inverse, then the inverse is denoted by $f^{-1} : B \to A$.

Remark 5.

1. $f^{-1} \circ f = 1_A$ and $(f^{-1} \circ f)(a) = a$, $\forall a \in A$.
2. $f \circ f^{-1} = 1_B$ and $(f \circ f^{-1})(b) = b$, $\forall b \in B$.
3. Given the graph of a function $f : A \to B$, where $A, B \subseteq \mathbb{R}$, the inverse function f^{-1} can be plotted by reflecting the graph of f in the line $x = y$. This is same as first reflecting f in *y-axis* followed by 90° clockwise rotation.