Lecture 8: Equivalence Relations

1 Equivalence Relations

Next interesting relation we will study is equivalence relation.

Definition 1.1 (Equivalence Relation). Let A be a set and let $\sim$ be a relation on A. The relation $\sim$ is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Example 1.2. Following are some of the examples of equivalence relations.

1. Let P be set of all people. Then, we can define an equivalence relation $\sim = \{(x, y) \in P \times P : x \text{ and } y \text{ have same age}\}$.

2. We can define an equivalence relation $\sim$ on real numbers as $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y\}$.

3. Let A be a non-empty set. We can define an equivalence relation $\sim$ on power set of A as $\{(S, T) \in \mathcal{P}(A) \times \mathcal{P}(A) : S = T\}$.

Definition 1.3. Let A be a non-empty set and $\sim$ be an equivalence relation on A.

1. The relation classes of A with respect to $\sim$ are called **equivalence classes**, and denoted $\sim [x]$ for all $x \in A$.

2. The **quotient set** of A and $\sim$ is the set $\{\sim [x] : x \in A\}$ of all equivalence classes of A with respect to $\sim$, and is denoted by $A/\sim$.

Example 1.4. Let P be the set of all people, and let $\sim$ be the relation on P defined by $x \sim y$ if and only if x and y are the same age (in years). If person x is 19 years old, then the equivalence class of x is the set of all 19-year olds. Each element of the quotient set $P/\sim$ is itself a set, where there is one such set consisting of all 1-year-olds, another consisting of all 2-year olds, and so on. Although there are billions of people in P, there are fewer than 125 elements in $P/\sim$, because no currently living person has reached the age of 125.

Theorem 1.5. *Let A be a non-empty set, and let $\sim$ be an equivalence relation on A.*

i) Let $x, y \in A$. If $x \sim y$, then $[x] = [y]$. If $x \nsim y$, then $[x] \cap [y] = \emptyset$.

ii) $\bigcup_{x \in A} [x] = A$.

Proof. Let A be a non-empty set, and let $\sim$ be an equivalence relation on A.

i) Let $z \in [x]$. Then $z \sim x$ by assumption and $x \sim y$ by hypothesis. By transitivity of $\sim$, we have $z \sim y$, and hence the $z \in [y]$. Therefore, we conclude $[x] \subseteq [y]$. Similarly, we can show $[y] \subseteq [x]$.

   We prove the second part by contradiction. Let $x \nsim y$, and $z \in [x] \cap [y]$. Then, $z \sim x$ and $z \sim y$. By transitivity and symmetry of $\sim$, we get $x \sim y$. Therefore, we have a contradiction.

ii) By definition, $[x] \subseteq A$ for all $x \in A$. Hence, $\bigcup_{x \in A}[x] \subseteq A$. Now, let $q \in A$. Then, $q \sim q$ by reflexivity. Therefore, $q \in [q] \subseteq \bigcup_{x \in A} [x]$. Hence, $A \subseteq \bigcup_{x \in A} [x]$.

□

Corollary 1.6. *Let A be a non-empty set, let $\sim$ be an equivalence relation on A. Let $x, y \in A$. Then $[x] = [y]$ iff $x \sim y$.*

Since equivalence relations are disjoint for unrelated elements, quotient sets separates a set into disjoint union of equivalence classes.

Definition 1.7 (Partition). Let A be a non-empty set. A **partition** of A is a family $\mathcal{D}$ of non-empty subsets of A such that

1. if $P, Q \in \mathcal{D}$ and $P \neq Q$, then $P \cap Q = \emptyset$, and
2. $\bigcup_{P \in \mathcal{D}} P = A$.

Example 1.8. We look at some examples of partitions.

1. Let $\mathcal{D} = \{E, O\}$, where E and O are set of even and odd integers respectively. Then, $\mathcal{D}$ is a partition of integers $\mathbb{Z}$.
2. Collection of sets $\mathcal{C} = \{[n, n+1) : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$.
3. Collection of sets $\mathcal{G} = \{(n-1, n+1)\}$ is *not* a partition of $\mathbb{R}$ because it is not pairwise disjoint. We have two sets (-1,1) and $(0, 2)$ in $\mathcal{C}$, that are not disjoint. In fact, $(-1, 1) \cap (0, 2) = (0, 1)$.

From Theorem 1.5, and definition of partitions, we have the following corollary.

Corollary 1.9. *Let A be a non-empty set, and let $\sim$ be an equivalence relation on A. Then $A/\sim$ is a partition of A.*

Definition 1.10. Let A be a non-empty set. Let $\mathcal{E}(A)$ denote the set of all equivalence relations on A. Let $\mathcal{T}_A$ denote the set of all partitions of A.

Example 1.11. Let $A = \{1, 2, 3\}$. Then $\mathcal{T}_A = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5\}$, where

$$\begin{aligned}
\mathcal{D}_1 &= \{\{1\}, \{2\}, \{3\}\}, \\
\mathcal{D}_2 &= \{\{1, 2\}, \{3\}\}, \\
\mathcal{D}_3 &= \{\{1, 3\}, \{2\}\}, \\
\mathcal{D}_4 &= \{\{2, 3\}, \{1\}\}, \\
\mathcal{D}_5 &= \{\{1, 2, 3\}\}.
\end{aligned}$$

Further, we see that $\mathcal{E}(A) = \{R_1, R_2, R_3, R_4, R_5\}$, where these equivalence relations are defined by the sets

$$\begin{aligned}
R_1 &= \{(1, 1), (2, 2), (3, 3)\}, \\
R_2 &= \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}, \\
R_3 &= \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}, \\
R_4 &= \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}, \\
R_5 &= \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}.
\end{aligned}$$

It is easy to see that each of the relations $\{R_i, i = 1, \ldots, 5\}$ listed above is an equivalence relation on A.

It is interesting to note that number of equivalence relations on a set, and number of partitions are same for the above example. It turns out that this is not mere coincidence.

Definition 1.12. Let A be a non-empty set. Let $\Phi : \mathcal{E}(A) \to \mathcal{T}_A$ be defined as follows. If $\sim$ is an equivalence relation on A, let $\Phi(\sim)$ be the family of sets $A/\sim$. Let $\Psi : \mathcal{T}_A \to \mathcal{E}(A)$ be defined as follows. If $\mathcal{D}$ is a partition of A, let $\Psi(\mathcal{D})$ be the relation on A defined by $x \; \Psi(\mathcal{D}) \; y$ iff there is some $P \in \mathcal{D}$ such that $x, y \in P$ for all $x, y \in A$.

Lemma 1.13. *Let A be a non-empty set. The functions Φ and Ψ in the above definition are well defined.*

Proof. To prove the lemma, we need to show the following two things.

1. For any equivalence relation $\sim$ on A, the family of sets $\Phi(\sim)$ is a partition of A.

2. For any partition $\mathcal{D}$ of A, the relation $\Psi(\mathcal{D})$ is an equivalence relation on A.

First part follows from the definition of Φ and Corollary 1.9. We will show the second part. Any partition $\mathcal{D} = \{P_i : i \in \{1, \ldots, n\}\}$ for some n. From definition of partition, it implies $A = \bigsqcup_{i=1}^{n} P_i$ and $P_i \cap P_j = \phi$ for all $i \neq j$. Therefore,

$$\Psi(\mathcal{D}) = \{(x, y) : \text{ there is some } i \in \{1, \ldots, n\} \in \mathcal{D} \text{ such that } x, y \in P_i\}.$$

We will show that $\Psi(\mathcal{D})$ is an equivalence relation on A.

Symmetry: Let $x\,\Psi(\mathcal{D})\,y$. Then we can find $P \in \mathcal{D}$ such that $x, y \in P$. Hence, $y\,\Psi(\mathcal{D})\,x$.

Reflexivity: Let $x \in A$. Then $x \in P$ for some $P \in \mathcal{D}$. Hence, $x\,\Psi(\mathcal{D})\,x$.

Transitivity: Let $x\,\Psi(\mathcal{D})\,y$ and $y\,\Psi(\mathcal{D})\,z$. Then, $x, y \in P$ for some $P \in \mathcal{D}$ and $y, z \in Q$ for some $Q \in \mathcal{D}$. Since, $y \in P \cap Q$ and P and Q are disjoint for $P \neq Q$, it follows that $P = Q$. This implies $x, y, z \in P$. Hence, $x\,\Psi(\mathcal{D})\,z$.

□

Example 1.14. We look at some examples of equivalence classes and related functions Φ, and partitions and related function Ψ.

1. Let $\sim$ be the relation on $\mathbb{R}^2$ defined by

$$(x, y) \sim (z, w) \text{ iff } y - x = w - z, \text{ for all } (x, y), (z, w) \in \mathbb{R}^2.$$

It can be verified that $\sim$ is an equivalence relation. We want to describe the partition $\Phi(\sim)$ of $\mathbb{R}^2$. Let $(x, y) \in \mathbb{R}^2$. Then

$$\sim [(x, y)] = \{(z, w) \in \mathbb{R}^2 | w - z = y - x\}.$$

Let $c = y - x$, then

$$\sim [(x, y)] = \{(z, w) \in \mathbb{R}^2 | w = z + c\},$$

which is just a line in $\mathbb{R}^2$ with slope 1 and y-intercept c. Hence, $\Phi(\sim)$ is collection of all lines in $\mathbb{R}^2$ with slope 1.

2. Let $\mathcal{C} = \{[n, n+1) : n \in \mathbb{Z}\}$ be a partition of $\mathbb{R}$. Then, we have corresponding family of equivalence relation classes, defined as

$$\begin{aligned}\Psi(\mathcal{C}) &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : x, y \in [n, n+1) \text{ for some } n \in \mathbb{Z}\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \lfloor x \rfloor = \lfloor y \rfloor\}.\end{aligned}$$

3. For partitions $\mathcal{D}_i$ and equivalence relations R_i defined in Example 1.11, we have $\Phi(R_i) = \mathcal{D}_i$ and $\Psi(\mathcal{D}_i) = R_i$ for all $i \in \{1, \ldots, 5\}$.

Theorem 1.15. *Let A be a non-empty set. Then the functions Φ and Ψ are inverses of each other, and hence both are bijective.*

Proof. We need to show that $\Psi \circ \Phi = 1_{\mathcal{E}(A)}$ and $\Phi \circ \Psi = 1_{\mathcal{T}_A}$. We will do this in following steps.

1. First, we prove that $\Psi \circ \Phi = 1_{\mathcal{E}(A)}$. Let $\sim \in \mathcal{E}(A)$ be an equivalence relation on A. Let $\approx = \Psi(\Phi(\sim))$. We will show that $\approx = \sim$, and it will then follow that $\Psi \circ \Phi = 1_{\mathcal{E}(A)}$. We show two relations are equal by set equality. Let $\mathcal{D} = \Phi(\sim)$, so that $\approx = \Psi(\mathcal{D})$.

   (a) First, we show that $\approx \subseteq \sim$. Let $x, y \in A$ such that $x \approx y$. Then, by the definition of Ψ there is some $P \in \mathcal{D}$ such that $x, y \in P$. By the definition of Ψ, we know that P is an equivalence class of $\sim$, so that $P = \sim [q]$ for some $q \in A$. Then $q \sim x$ and $q \sim y$, and by the symmetry and transitivity of $\sim$ it follows that $x \sim y$. Hence, $\approx \subseteq \sim$.

   (b) Now, we show that $\sim \subseteq \approx$. Let $x \sim y$ in A. Then, $y \in [x]$. By the reflexivity of $\sim$, we know that $x \in \sim [x]$. From definition of Φ we have $\sim [x] \in \mathcal{D}$. Hence, by the definition of Ψ, it follows that $x \approx y$. Hence, $\sim \subseteq \approx$.

2. Second, we prove that $\Phi \circ \Psi = 1_{\mathcal{T}_A}$. Let $\mathcal{D} \in \mathcal{T}_A$ be a partition of A. Let $\mathcal{F} = \Phi(\Psi(\mathcal{D}))$. We will show that $\mathcal{F} = \mathcal{D}$, and it will then follow that $\Phi \circ \Psi = 1_{\mathcal{T}_A}$. Let $\sim = \Psi(\mathcal{D})$, so that $\mathcal{F} = \Psi(\sim)$.

   (a) Let $B \in \mathcal{F}$. Then by the definition of Φ we know that B is an equivalence class of $\sim$, so that $B = \sim [z]$ for some $z \in A$. Because $\mathcal{D}$ is a partition of A, then there is a unique $P \in \mathcal{D}$ such that $z \in P$. Let $w \in A$. Then by the definition of Ψ we see that $z \sim w$ if and only if $w \in P$. It follows that $w \in \sim [z]$ if and only if $w \in P$, and hence $P = \sim [z]$. Hence $B = \sim [z] = P \in \mathcal{D}$. Therefore $\mathcal{F} \subseteq \mathcal{D}$.

   (b) Let $C \in \mathcal{D}$. Let $y \in C$. As before, it follows from the definition of Ψ that $C = \sim [y]$. Therefore, by the definition of Φ we see that $C \in \Phi(\sim) = \mathcal{F}$. Hence, $\mathcal{D} \subseteq \mathcal{F}$.

□