Lecture 10: Recursion

1 Recursion

Consider a sequence $\{1, 2, 4, 8, 16, \ldots\}$. This sequence is described in two ways. First way is, let a_n denote the n^{th} term of the sequence, then $a_n = 2^{n-1}$, $\forall n \in \mathbb{N}$. Second way is, let $a_1 = 1$, and $a_{n+1} = 2a_n$, $\forall n \in \mathbb{N}$. Such a description is called a "recursive" description of the sequence.

Given a sequence for which we already have an explicit formula for each a_n in terms of n, it can be useful to find a recursive formula, but there is no question that the sequence exists. What about a sequence for which we have only a recursive description, but no explicit formula?

Example 1.1. Suppose that we have the recursive description $c_1 = 4$, and $c_{n+1} = 3 + 2c_n$ for all $n \in \mathbb{N}$. Is there a sequence $\{c_1, c_2, c_3, \ldots\}$ satisfying such a description? That is, does this description actually define a sequence?

Remark 1. Intuitively, it seems that there exists such a sequence, because we can proceed "inductively", producing one element at a time. We know that $c_1 = 4$. We can then compute $c_2 = 3+2c_1 = 3+2\times4 = 11$, and $c_3 = 3+2c_2 = 3+2\times11 = 25$, and so on.

There are a number of variations of the process of definition by recursion, the most basic of which is as follows. Suppose we are given a number $b \in X$, and a function $h : X \to X$. We then want to define a sequence $\{a_1, a_2, \ldots\} \subseteq X$ such that $a_1 = b$ and that $a_{n+1} = h(a_n)$, for all $n \in \mathbb{N}$. We can now state the theorem that guarantees the validity of definition by recursion.

Theorem 1.2 (Definition by Recursion). *Let A be a set, $b \in A$, and $k : A \to A$ be a function. Then there is a unique function $f : \mathbb{N} \to A$ such that $f(1) = b$, and that $f(n+1) = k(f(n))$ for all $n \in \mathbb{N}$.*

Remark 2. Informally, definition by recursion says that if A is a set, $b \in A$, and $k : A \to A$ is a function, then there is a unique sequence $\{a_n : n \in \mathbb{N}\} \subseteq A$ such that $a_1 = b$, and that $a_{n+1} = k(a_n)$ for all $n \in \mathbb{N}$.

Example 1.3. In previous example we were looking for existence of sequence $\{c_n : n \in \mathbb{N}\}$ such that $c_1 = 4$, and $c_{n+1} = 3 + 2c_n$ for all $n \in \mathbb{N}$. Let $b = 4$, and function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = 3 + 2x$ for all $x \in \mathbb{R}$. By definition by recursion, we have a unique function $f : \mathbb{N} \to \mathbb{R}$ such that $f(1) = 4$ and $f(n + 1) = 3 + 2 \cdot f(n)$ for all $n \in \mathbb{N}$. If we let $c_n = f(n)$ for all $n \in \mathbb{N}$, then the sequence $\{c_n : n \in \mathbb{N}\}$ satisfies the recursive conditions.

Remark 3. We make some observations about definition by recursion.

1. Definition by recursion gives us existence of a unique sequence with the desired properties, it does not give us an explicit formula for this sequence.
2. Usual techniques to find explicit formulas are generating functions, which unfortunately are out of scope of these lectures.
3. In case of Example 1.3, we can guess the formula $c_n = 7 * 2^{n-1}$ for all $n \in \mathbb{N}$, and use PMI to show this formula holds.
4. Let A be a non-empty set, and let $f : A \to A$ be a function. For any $n \in \mathbb{N}$, we would like to define a function denoted f^n, by the formula

$$f^n = \underbrace{f \circ \ldots \circ f}_{n \text{ times}}$$

Definition 1.4. Fix a function $f \in \mathcal{F}(A, A)$. Then, we can define a function $k : \mathcal{F} \to \mathcal{F}$ by $k(g) = f \circ g$ for all $g \in \mathcal{F}(A, A)$. By definition by recursion applied to the set $\mathcal{F}(A, A)$, the element $f \in \mathcal{F}(A, A)$, and the function $k : \mathcal{F}(A, A) \to \mathcal{F}(A, A)$, there exists a unique function $\phi : \mathbb{N} \to \mathcal{F}(A, A)$ such that $\phi(1) = f$ and $k \circ \phi = f \circ \phi = \phi \circ s$. We define f^n as $\phi(n)$ for all $n \in \mathbb{N}$, and refer to it as the **n-fold iteration of f**.

Remark 4. We make the following observations about this n-fold iteration of f.

1. Notice that $f^1 = \phi(1) = f$, and as we expect for all $n \in \mathbb{N}$, we have

$$f^{n+1} = \phi(n + 1) = (\phi \circ s)(n) = (f \circ \phi)(n) = f \circ f^n.$$

2. It's easier to understand this definition by the commutative diagram in Figure 1.

We have a variant of definition by recursion theorem by defining $a_{n+1} = k(n, a_n)$.

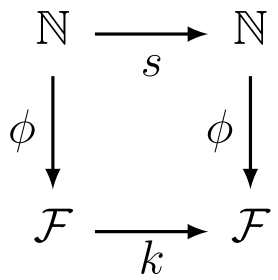

Figure 1: Commutative diagram for recursive function $k : \mathcal{F} \to \mathcal{F}$, where $\mathcal{F}$ is shorthand for space of functions $\mathcal{F}(A, A)$.

Theorem 1.5. *Let A be a set, $b \in A$, and $t : A \times \mathbb{N} \to A$ be a function. Then there is a unique function $g : \mathbb{N} \to A$ such that $g(1) = b$, and $g(n+1) = t(g(n), n)$ for all $n \in \mathbb{N}$.*

Proof. We apply Theorem 1.2 to set $B = A \times \mathbb{N}$, element $(b, 1) \in B$, function $\tilde{t} : B \to B$, to get a unique function $\tilde{g} : \mathbb{N} \to B$ such that $\tilde{g}(1) = (b, 1)$ and $\tilde{g} \circ s = \tilde{t} \circ \tilde{g}$. Let $\pi_1 : B \to A$ and $\pi_2 : B \to \mathbb{N}$ be coordinate-wise projections of elements in B to A and $\mathbb{N}$ respectively, such that $\pi_1(a, n) = a$ and $\pi_2(a, n) = n$ for all $(a, n) \in B$. Let $\tilde{t} = (t, s \circ \pi_2)$ for all elements in B, then we have $\tilde{g}(1) = (b, 1)$, and

$$\tilde{g} \circ s = \tilde{t} \circ \tilde{g} = (t \circ \tilde{g}, s \circ \pi_2 \circ \tilde{g}).$$

We let $g = \pi_1 \circ \tilde{g}$. Then, we observe that $g(1) = b$, and

$$g \circ s = \pi_1 \circ \tilde{g} \circ s = \pi_1 \circ \tilde{t} \circ \tilde{g} = t \circ \tilde{g} = t \circ (g, \pi_2 \circ \tilde{g}).$$

It suffices to show that $(\pi_2 \circ \tilde{g})(n) = n$ holds for all $n \in \mathbb{N}$. We can show this by induction on $n \in \mathbb{N}$. It is clear that $(\pi_2 \circ \tilde{g})(1) = 1$. Assuming that inductive hypothesis holds for n, and observing that $(\pi_2 \circ \tilde{g} \circ s) = (s \circ \pi_2 \circ \tilde{g})$, we see that it is indeed true. □

Example 1.6. We have few interesting examples of such recursions.

1. We define the following recursion $a_1 = 1$, and $a_{n+1} = (n+1)a_n$ for all $n \in \mathbb{N}$. This recursion looks curiously like familiar sequence of factorials. We formally show existence of such a sequence by defining a function $t : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ as $t(x, m) = (m+1)x$ for all $(x, m) \in \mathbb{R} \times \mathbb{N}$. Applying Theorem 1.5 for $A = \mathbb{R}$, $b = 1$, and function t, we see that there is a unique sequence $\phi \in \mathbb{N}, \mathbb{R}$ satisfying these conditions. Notice that $\phi(1) = 1$, and $\phi(n+1) = t(\phi(n), n) = (n+1)\phi(n)$. We denote $\phi(n)$ by $n!$.

2. Let $f : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence, and $q : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ be a function defined as $q(x, n) = x + f(n + 1)$. Then, the recursive definition $a_1 = f(1)$ and $a_{n+1} = q(a_n, n)$ leads to a unique sequence $h : \mathbb{N} \to \mathbb{R}$ using Theorem 1.5 for $A = \mathbb{R}$, $b = f(1)$, and function q, satisfying recursive conditions.

3. We would like to further generalize recursive equations when $a_{n+1} = f(a_n, a_{n-1})$.

Theorem 1.7. *Let A be a set, let $a, b \in A$ and let $p : A \times A \to A$ be a function. Then there is a unique function $f : \mathbb{N} \to A$ such that $f(1) = a$, $f(2) = b$, and $f(n + 2) = p(f(n), f(n + 1))$ for all $n \in \mathbb{N}$.*

Proof. We apply Theorem 1.2 to set $B = A \times A$, element $(a, b) \in B$, function $\tilde{p} : B \to B$, to get a unique function $\tilde{f} : \mathbb{N} \to B$ such that $\tilde{f}(1) = (a, b)$ and $\tilde{f} \circ s = \tilde{p} \circ \tilde{f}$. Let π_1 and π_2 be coordinate-wise projections of elements in B to A, such that $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all $(x, y) \in B$. Let $\tilde{p} = (\pi_2, p)$ for all elements in B, then we have $\tilde{f}(1) = (a, b)$, and

$$\tilde{f} \circ s = \tilde{p} \circ \tilde{f} = (\pi_2 \circ \tilde{f}, p \circ \tilde{f}).$$

Let $f = \pi_1 \circ \tilde{f}$. Then, we observe that $f(1) = a$, and

$$f \circ s = \pi_1 \circ \tilde{f} \circ s = \pi_1 \circ \tilde{p} \circ \tilde{f} = \pi_2 \circ \tilde{f},$$
$$f \circ s \circ s = \pi_2 \circ \tilde{f} \circ s = p \circ \tilde{f} = p \circ (f, \pi_2 \circ \tilde{f}) = p \circ (f, f \circ s).$$

□

Definition 1.8. Let $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined by $p(x, y) = x + y$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Then, the **Fibonacci sequence** is the unique sequence denoted $\{F_n : n \in \mathbb{N}\}$ defined by $F_1 = 1, F_2 = 2$ and $F_{n+2} = p(F_n, F_{n+1})$.

Remark 5. Notice the existence of this unique sequence follows from Theorem 1.7 applied to the case when $A = \mathbb{R}, a = 1, b = 1$, and function $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $p(x, y) = x + y$.

The following proposition gives a few examples of formulas involving the sums and products of Fibonacci numbers.

Proposition 1.9. *Let $n \in \mathbb{N}$. Then the following are true for Fibonacci sequence $\{F_n : n \in \mathbb{N}\}$.*

1. $F_1 + F_2 + ... + F_n = F_{n+2} - 1$.

2. $F_1^2 + F_2^2 + ... + F_n^2 = F_n F_{n+1}$.

3. *If $n \geq 2$, then* $(F_n)^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$.

Proof. We will show using induction, and the fact that $F_n + F_{n+1} = F_{n+2}$.

1. We use induction on n. We see that $F_1 = 1 = 1+1-1 = F_1 + F_2 - 1$, hence the hypothesis holds for $n = 1$. Assuming inductive hypothesis for n, we see that

$$\sum_{i=1}^{n+1} F_i = \sum_{i=1}^{n} F_i + F_{n+1} = F_{n+1} + F_{n+2} - 1 = F_{n+3} - 1.$$

2. We use induction on n. We see that $F_1^2 = 1 = 1.1 = F_1 F_2$, hence the hypothesis holds for $n = 1$. Assuming inductive hypothesis for n, we see that

$$\sum_{i=1}^{n+1} F_i^2 = \sum_{i=1}^{n} F_i^2 + F_{n+1}^2 = F_n F_{n+1} + F_{n+1}^2 = F_{n+1} F_{n+2}.$$

3. We use PMI-V3 with $k_0 = 2$. We see that $(F_2)^2 - F_3 F_1 = 1^2 - 2.1 = -1 = (-1)^{2+1}$, so the equation holds for $n = 2$. Now let $n \in \mathbb{N}$. Suppose that $n \geq 3$, and that the equation holds for all values in $\{2, ..., n\}$. We compute

$$\begin{aligned}(F_{n+1})^2 - F_{n+2}F_n &= (F_{n+1})^2 - (F_{n+1} + F_n)F_n = F_{n+1}(F_{n+1} - F_n) - F_n^2 \\ &= -(F_n^2 - F_{n+1}F_{n-1}) = -(-1)^{n+1} = (-1)^{n+2}.\end{aligned}$$

where the last line holds by the inductive hypothesis.

□

Remark 6. Binet's formula for explicitly writing Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}} \left[\phi^n - (-\phi^{-1})^n\right], \; \forall n \in \mathbb{N},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Definition 1.10. Let A be a set. Let $\mathcal{G}(A)$ be the set defined by

$$\mathcal{G}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{F}([n], A).$$

Theorem 1.11. *Let A be a non-empty set, an element $b \in A$, and $k : \mathcal{G}(A) \to A$ be a function. Then there is a unique function $f : \mathbb{N} \to A$ such that $f(1) = b$, and that $f(n+1) = k(f|_{[n]})$ for all $n \in \mathbb{N}$.*

Proof. As usually is the case, uniqueness is easier to show than existence.

Uniqueness: Let $s, t : \mathbb{N} \to A$ be functions. Suppose that $s(1) = t(1) = b$, and $s(n+1) = k(s|_{[n]})$ and $t(n+1) = k(t|_{[n]})$ for all $n \in \mathbb{N}$. We will show that $s(n) = t(n)$ for all $n \in \mathbb{N}$ by induction on n, using PMI-V2.

By hypothesis, we know that $s(1) = t(1) = b$. Next, let $n \in N$ and suppose that $s(j) = t(j)$ for all $j \in [n]$. Then $s|_{[n]} = t|_{[n]}$, and therefore $s(n+1) = k(s|_{[n]}) = k(t|_{[n]}) = t(n+1)$. It now follows from PMI-V2 that $s(n) = t(n)$ for all$n \in \mathbb{N}$, which means that $s = t$.

Existence: There are three steps in the definition of f.

Step 1. We will shown that for each $p \in \mathbb{N}$, there is a function $h_p : [p] \to A$ such that $h_p(1) = b$, and that $h_p(n+1) = k(h_p|_{[n]})$ for all $n \in [p-1]$. The proof is by induction on p. First, let $p = 1$. Then, $[p] = \{1\}$. Let $h_1 : [1] \to A$ be defined by $h_1(1) = b$. Observe that $[p-1] = \emptyset$, and hence $h_1(n+1) = k(h_1|_{[n]})$ for all $n \in [p-1]$ is necessarily true. Next, let $p \in \mathbb{N}$, and assume that inductive hypothesis is true for p, for a function $h_p : [p] \to A$. We define a function $h_{p+1} : [p+1] \to A$ as

$$h_{p+1}(n) = \begin{cases} h_p(n), & n \in [p], \\ k(h_p), & n = p+1. \end{cases}$$

Then $h_{p+1}|_{[p]} = h_p$. It follows that $h_{p+1}(1) = h_p(1) = b$, and

$$h_{p+1}(n+1) = h_p(n+1) = k(h_p|_{[n]}) = k(h_{p+1}|_{[n]}) = h_p(n) \forall n \in [p-1],$$

and that $h_{p+1}(p+1) = k(h_p) = k(h_{p+1}|_{[p]})$. Hence h_{p+1} has the desired properties. The proof of this step is then complete by PMI.

Step 2. Let $p, q \in \mathbb{N}$. Suppose that $p < q$. We can show that $h_q(n) = h_p(n)$ for all $n \in [p]$. By Step 1, we know that $h_q(1) = h_p(1) = b$. Next, suppose that $n \in [p-1]$ and that $h_q(j) = h_p(j)$ for all $j \in [n]$. Hence $h_q|[n] = h_p|_{[n]}$. Then by Step 1 we see that $h_q(n+1) = k(h_q|_{[n]}) = k(h_p|_{[n]}) = h_p(n+1)$. It now follows that $h_q(n) = h_p(n)$ for all $n \in [p]$.

Step 3. Let $f : \mathbb{N} \to A$ be defined by $f(n) = h_n(n)$ for all $n \in \mathbb{N}$. Then $f(1) = h_1(1) = b$ by Step 1. Let $p \in \mathbb{N}$. If $j \in [p]$, then $j < p+1$, and it follows from Step 2 that $h_{p+1}(j) = h_j(j) = f(j)$. Hence $h_{p+1}|_{[p]} = f|_{[p]}\}$. Using Step 1 we then see that $f(p+1) = h_{p+1}(p+1) = k(h_{p+1}|_{[p]}) = k(f|_{[p]})$.

We therefore see that f satisfies the desired properties.

□