Lecture 14: The Order Topology

1 The Order Topology

If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called "Order Topology".

Definition: Suppose X is a set having order relation <. Given a<b ϵ X, there are four subsets of X called intervals determined by a and b. They are as follows :

$$(a, b) = \{x \mid a < x < b\}$$
$$(a, b] = \{x \mid a < x \leq b\}$$
$$[a, b) = \{x \mid a \leq x < b\}$$
$$[a, b] = \{x \mid a \leq x \leq b\}$$

Definition: Let X be a simple order relation.Assume that X has more than one element.Let $\mathcal{B}$ be the collection of all sets of the following types:

1.All open intervals (a,b) in X.
2.All intervals of the form [a_0,b), where a_0 is the smallest element (if any) of X.
3.All intervals of the form (a,b_0], where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology.

Remark: If X has no smallest then there are no sets of type **2** and if X has no largest element then there are no sets of type **3**.

Verification that $\mathcal{B}$ satisfies the requirements for being a basis: Let x ϵ X. If x=a_0 then x ϵ[a_0,b).If x=b_0, then x ϵ (a,b_0].These 2 cases are easy to verify. But let us consider the scenario where x$\neq$ a_0 and x$\neq$ b_0.

From definition, $(a, b) = \{z \mid a < z < b\}$. As the order relation ¡ is defined on the set X, there will some element a ϵ X and some element b ϵ such that a¡x¡b.In that case a ϵ (a,b). Hence we have verified that $\mathcal{B}$ satisfies the first criteria for being a basis over the order topology on X.

Now we need to check for the 2nd which $\mathcal{B}$ B needs to satisfy for being a basis over the order topology on X. Let us take $B_1 = [a_0, b)$,$B_2 = (a, b_0]$, $B_3 = (a_1, b_1)$ and $B_4 = (a_2, b_2)$, where a_0 is the smallest element in X and b_0 is the largest element in X. The sets $B_1, B_2, B_3.B_4$ all belong to $\mathcal{B}$. If x $\epsilon B_1 \cap B_2$, then x $\epsilon(a,b)$. The set (a,b) is thus a subset of $B_1 \cap B_2$. Hence, x $\epsilon(a,b) \subset B_1 \cap B_2 \epsilon \mathcal{B}$. Thus if X has both largest and smallest element, then $\mathcal{B}$ satisfies the criteria for being a basis.

We shall now inspect the case that X has no largest element.If x $\epsilon B_1 \cap B_3$. Depending on the value of b,a_1 and b_1, we will be able to a set of the form (p,q), such that x $\epsilon(p,q) \subset B_1 \cap B_3 \epsilon \mathcal{B}$. Thus if X has no largest element but a smallest element, then $\mathcal{B}$ satisfies the criteria for being a basis.

Let us consider the case that X has no smallest element. If x $\epsilon B_2 \cap B_3$. Depending on the value of a,a_1 and b_1, we will be able to a set of the form (r,s), such that x $\epsilon(r,s) \subset B_2 \cap B_3 \epsilon \mathcal{B}$. Thus if X has no largest element but a smallest element, then $\mathcal{B}$ satisfies the criteria for being a basis.

Finally, we will consider the case when X has no largest or smallest element. So the basis of the order topology will consist of sets of the form (m,n). If x $\epsilon B_3 \cap B_4$. Depending on the value of a_1,b_1,a_2 and b_2, we will be able to a set of the form (e,d), such that x $\epsilon(e,d) \subset B_3 \cap B_4 \epsilon \mathcal{B}$. Thus if X has no largest and smallest element, then $\mathcal{B}$ satisfies the criteria for being a basis.

Thus we have considered all possible scenarios which should be checked for $\mathcal{B}$ for being a basis for the order topology on X. Since $\mathcal{B}$ has satisfied all of them, it is a basis for order topology on X.

Example 1: The standard topology on $\mathbb{R}$ is just the order topology derived from the usual order on $\mathbb{R}$.

Example 2: Consider the set $\mathbb{R} \times \mathbb{R}$ in the dictionary order; we shall denote the general element of $\mathbb{R} \times \mathbb{R}$ by x×y, to avoid difficulty with notation. The set $\mathbb{R} \times \mathbb{R}$ has neither a largest nor a smallest element, so the order topology on $\mathbb{R} \times \mathbb{R}$ has as basis the collection of all open intervals of the form (a×b, ×d) for a<c,and when a equals c,b<d.

Example 3: The positive integers $\mathbb{Z}_+$ form an ordered set with a smallest element. The order topology on $\mathbb{Z}_+$, is the discrete topology,as the singletons are open sets. If n>1,then the singleton $\{n\}$=(n-1,n+1) is a basis element;and if n=1,the one-point set $\{1\}$=[1,2) is a basis element.

Example 4: The set X=$\{1,2\} \times \mathbb{Z}_+$ in the dictionary order is another example of an ordered set with a smallest element. Denoting 1×n by a_n and 2×n by b_n, we can represent X by $\{a_1, a_2....., b_1, b_2......\}$ The order topology on X is not the discrete topology. Most one-point sets are open, but there is an exception .The one-point set $\{b_1\}$. Any open set containing b_1 must contain a basis element about b_1 (by definition),but any basis element containing b_1 contains points of the a_i sequence.

Definition:If X is an ordered set and a is an element of X,there are four subsets of X that are called the rays determined by a. They are the following:

$$(-\infty, a) = \{x \mid x < a\}$$
$$(a, +\infty) = \{x \mid x > a\}$$
$$(-\infty, a] = \{x \mid x \leq a\}$$
$$[a, +\infty) = \{x \mid x \geq a\}$$

The sets of type $(-\infty, a)$ and $(a, +\infty)$ are called open rays. Similarly, the sets of type $(-\infty, a]$ and $[a, +\infty)$ are called closed rays.
We now need to verify that the open rays belong to the order topology on X. We shall first consider the open ray $(a, +\infty)$.
If X contains a largest element b_0 then $(a, +\infty)$ is of the form $(a, b_0]$,which is a basis element for the order topology on X. Thus in this case $(a, +\infty)$ is open. If X has no largest element, then $(a, +\infty)$=$\bigcap_{x>a}(a, x)$. To prove this statement, let z $\epsilon(a, +\infty)$. From definition, z> a. Hence if we consider the set $\bigcap_{x>a}(a, x)$, z will belong at least one of the subsets of the form (a,x). Thus z $\epsilon \bigcap_{x>a}(a, x)$. Hence $(-\infty, a) \subset \bigcap_{x>a}(a, x)$.
Now let p $\epsilon \bigcap_{x>a}(a, x)$. So p will belong to at least one of the open intervals of the form (a, x). If p belongs to some open interval of the form (a, x), then from definition $a < p < x$. From definition of the open rays, p $\epsilon(a, +\infty)$. Thus $bigcap_{x>a}(a, x) \subset (-\infty, a))$. Also, we have proven that $(-\infty, a) \subset \bigcap_{x>a}(a, x)$. So $(-\infty, a) = \bigcap_{x>a}(a, x)$.
We shall now consider the open ray $(-\infty, a)$.If X contains a smallest element a_0 then $(-\infty, a))$ is of the form $[a_0, a)$,which is a basis element for the order topology on X.Thus in this case $(-\infty, a)$ is open.If X has no smallest element, then $(-\infty, a)$=$\bigcap_{x<a}(x, a)$.To prove this statement, let w $\epsilon(-\infty, a)$.From definition, w< a. Hence if we consider the set $\bigcap_{x<a}(x, a)$, w will belong at least one of the subsets of the form (x,a). Thus w $\epsilon \bigcap_{x<a}(x, a)$. Hence $(-\infty, a) \subset \bigcap_{x<a}(x, a)$.
Now let q $\epsilon \bigcap_{x<a}(x, a)$. So q will belong to at least one of the open intervals of the form (x, a). If p belongs to some open interval of the form (x, a), then from definition $x < q < a$. From definition of the open rays, q $\epsilon(-\infty, a)$. Thus $\bigcap_{x<a}(x, a) \subset (-\infty, a))$. Also, we have proven that $(-\infty, a) \subset \bigcap_{x<a}(x, a)$. So $(-\infty, a) = \bigcap_{x<a}(x, a)$.
Hence both the open rays belong to the order topology on X.

Remark: The open rays form a sub-basis for the order topology on X.

Proof:The open rays $(-\infty, a)$ and $(a, +\infty)$ are open sets in the order topology defined on X. Hence the topology generated by $(-\infty, a)$ and $(a, +\infty)$ are contained in the order topology on X. If $\mathfrak{T}_R$ be the topology generated by the open intervals and if $\mathfrak{T}$ be the order topology on X, then we write $\mathfrak{T}_R \subset \mathfrak{T}$.

If we consider the intersection of the open rays of the form $(-\infty, b)$ and $(a, +\infty)$, then it is the open interval of the form (a,b).The set (a,b) is a basis element of the order topology on X. If X has a smallest element a_0, then $(-\infty, b))$ is of the form $[a_0, b)$. Then the intersection of $[a_0, b)$ with $(a, +\infty)$, will yield an interval of the form (a,b), which is a basis element for the order topology on X.

Similarly,if X has a largest element b_0, then (a, ∞) is of the form $(a, b_0]$. Then the intersection of $(a, b_0]$ with $(-\infty, b)$, will yield an interval of the form (a,b), which is a basis element for the order topology on X. For both the largest element and the smallest element cases, we have assumed that the intersection between the sets is non-empty. If it is empty, then the basis elements of the form $(a, b_0]$ or $[a_0, b)$ which both again are subsets of the order topology on X. Thus, finite intersection of the open rays yield the basis elements for the order topology on X. Also, X= $(-\infty, a) \cap (a, +\infty)$. Hence the open rays satisfy the criteria for being a sub-basis for the order topology on X.

2 The Product Topology

Definition: Let X and Y be topological spaces. The product topology on X×Y is the topology having as basis the collection $\mathcal{B}$ of all sets of the form U×V where U is an open set in X and V is an open set in Y.

We need to check whether $\mathcal{B}$ is a basis over X×Y.Let (x,y)ϵ X×Y.The collection $\mathcal{B}$ contains elements of the form U ×V, where U and V are open sets in X and Y respectively. So U ϵ X and V ϵ Y.The element (x,y) belongs to the product topology on X×Y. So there must be some U ϵ X and V ϵ Y such that X ϵ X and y ϵ V. Thus (x,y) ϵ U×V $\subset$ X×Y. Now U×V ϵ $\mathcal{B}$.So the elements of the set $\mathcal{B}$ satisfy the first criteria for being a basis of the product topology on X×Y.

Let us take $B_1 \epsilon \mathcal{B}$ and $B_2 \epsilon \mathcal{B}$ such that B_1=U×V and B_2=T×W.The sets U and T are open in X and the sets V and W are open in Y.So we can write $B_1 \cap B_2$=(U×V)∩(T×W).Now $B_1 \cap B_2$ can be also written as (U∩T)×(V∩W).If (a,b) ϵ $B_1 \cap B_2$, then (a,b) ϵ (U∩T)×(V∩W).Since the sets U and T are open in X and the sets V and W are open in Y,so (U∩ T) and (V∩ W) are open in X and Y respectively. Let U_0=(U∩ T) and V_0=(V∩ W). Thus we have (a,b)ϵ $(U_0 \cap V_0) \subset$ (U∩ T)×(V∩ W).Also $(U_0 \cap V_0)$ $\epsilon \mathcal{B}$. Thus the elements of $\mathcal{B}$ satisfy the two necessary conditions for being a basis of the product topology on X×Y. Hence the elements of $\mathcal{B}$ form a basis.

Theorem 15.1: If $\mathcal{B}$ be the basis for a topology on X and $\mathcal{C}$ be the basis for a topology on Y, then,

$$\mathcal{D} = \{B \times C \mid B \epsilon \mathcal{B} \text{ and } C \epsilon \mathcal{C}\}$$

is the basis for the topology on X×Y.

Proof:Let us consider the element W×T which belongs to the product topology on X×Y. By definition of product topology, there exists an element U×V in X×Y such that (x,y)ϵ U×V $\subset$ X×Y. The set U×V is then an basis element for the product topology on X×Y and U×V $\epsilon\mathcal{B}$.So,(x,y)ϵ U×V, xϵU and yϵV. The sets $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y respectively.Hence we can find B ϵ $\mathcal{B}$ and Cϵ $\mathcal{B}$ such that xϵ B$\subset$U and yϵ C$\subset$V. So we can write (x,y)ϵ B×C $\subset$ W×T. If $\mathcal{D} = \{B \times C \mid B\epsilon\mathcal{B} \text{ and } C\epsilon\mathcal{C}\}$, then from Lemma 13.2, $\mathcal{D}$ meets the criteria for being a basis for the product topology on X×Y.

Example: Let us consider the order topology on $\mathbb{R}$. The product of this topology with itself is called the standard topology on $\mathbb{R} \times \mathbb{R}$. Now we can write $\mathbb{R} \times \mathbb{R}$ as $\mathbb{R}^2$.By definition, the collection of all sets of the form U×V, where U of the form (p,q) and V of the form (r,s),form the basis for the product topology on $\mathbb{R}^2$. By theorem 15.1, the basis for the product topology on $\mathbb{R}^2$ can also be represented by,

$$\mathcal{D} = \{B \times C \mid B\epsilon\mathbb{R} \text{ and } C\epsilon\mathbb{R}\}.$$

In the above both B and C are basis elements of $\mathbb{R}$ and is of the form (a,b).

Definition: Let π_1:X×Y→X be defined by the,

$$\pi_1\text{(x,y)=x.}$$

Also,let π_2:X×Y→Y be defined by the equation.

$$\pi_1\text{(x,y)=y.}$$

The maps π_1 and π_2 are called the projections of X×Y onto its first and second factors respectively. The word "onto" is used here because the mapping is subjective.

Let X and Y be topological spaces.Let us consider the product topology on X×Y. Assume, U$\subset$X and V$\subset$Y. Then we have the following,

$$\textbf{(a)}\pi_1\text{(u,y)=u and } \pi_1^{-1}\text{(u)=(u,y).}$$
$$\textbf{(b)}\pi_2\text{(x,v)=v and } \pi_2^{-1}\text{(v)=(x,v).}$$

Theorem 15.2: The collection,

$$\mathcal{S}=\{\pi_1^{-1}(\text{U})|\ \text{U}\epsilon\text{X}\} \cup \{\pi_2^{-1}(\text{V})|\ \text{v}\epsilon\text{Y}\},$$

is a sub-basis for product topology on X×Y.

Proof: Let $\mathcal{T}_S$ be the topology generated by $\mathcal{S}$ and let $\mathcal{T}$ be the product topology on X×Y. Each and every element of $\mathcal{S}$ belongs to $\mathcal{T}$. So the arbitrary unions of finite intersections of the elements of $\mathcal{S}$ also belong to $\mathcal{T}$. Hence $\mathcal{T}_S \subset \mathcal{T}$.

On the other hand, all the sets of the form U×V, where U and V are open in X and Y respectively, form the basis for the product topology on X×Y. Since $\mathcal{S}$ is the sub-basis for the product topology, the union of the elements of $\mathcal{S}$ generates the entire set X×Y. Also, if we consider the finite intersection of the elements of $\mathcal{S}$, π_1^{-1}(U) $\cap\pi_2^{-1}$(V),then it will be equal to U×V. Thus U×V $\epsilon\mathcal{T}_S$. Hence $\mathcal{T} \subset \mathcal{T}_S$. So we have $\mathcal{T}$=$\mathcal{T}_S$.From this, we can infer that the topology generated by $\mathcal{S}$ is same as the product topology on X×Y.