Lecture 15: The subspace topology, Closed sets

1 The Subspace Topology

Definition 1.1. Let $(X, \mathcal{T})$ be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of alt intersections of open sets of X with Y.

Check that $\mathcal{T}_Y$ is a topology:

1. It contains ϕ and Y because

$$\phi = Y \cap \phi \, and \, Y = Y \cap X.$$

   where ϕ and X are elements of $\mathcal{T}$.

2. It is closed under finite intersections:

$$(U_1 \cap Y) \cap ... \cap (U_n \cap Y) = (U_1 \cap ... \cap U_n) \cap Y.$$

3. It is closed under arbitrary unions:

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = (\bigcup_{\alpha \in J} U_\alpha) \cap Y.$$

Lemma 1.2. *Let $\mathcal{B}$ be a basis for the topology of X then the collection*

$$\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Given U open in X and given $y \in U \cap Y$, we can choose an element B of $\mathcal{B}$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows from Lemma 13.2 that $\mathcal{B}_Y$ is a basis for the subspace topology on Y. □

When dealing with a space X and a subspace Y, one needs to be careful when one uses the term "open set". Does one mean an element of the topology of Y or an element of the topology of X? We make the following definition:

Definition 1.3. If Y is a subspace of X, we say that a set U is **open in** Y if $U \in \mathcal{T}_Y$; this implies in particular that it is a subset of Y. We say that U is **open in** X if $U \in \mathcal{T}_X$.

Lemma 1.4. *Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.*

Proof. Since U is open in Y, $U = Y \cap V$ for some set V open in X. Since Y and V are both open in X, so is $Y \cap V$. □

In the following theorem, the relation between the subspace topology and the order and product topologies has been discussed.

Theorem 1.5. *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y. Therefore, $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now,

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Since $U \cap A$ and $V \times B$ are the general open sets for the subspace topologies on A and B, respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$. Hence, we can conclude that the bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same. Hence the topologies are the same. □

Now, let X be an ordered set in the order topology, and let Y be a subset of X. The order relation on X, when restricted to Y, makes Y into an ordered set. However, *the resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X.*
Consider the following examples:

Example 1.6. Consider the subset $Y = [0,1] \subseteq \mathbb{R}$, in the subspace topology. The subspace topology has as basis $\mathcal{B} = \{(a,b) \cap Y | (a,b)\, is\, an\, open\, interval\, in\, \mathbb{R}\}$. Such a set is of one of the following types.

$$(a,b) \cap Y = \begin{cases} (a,b) & \text{if } a,b \in Y \\ [a,b) & \text{if } a \notin Y, b \in Y \\ (a,1] & \text{if } b \notin Y, a \in Y \\ Y\, or\, \phi & \text{if } b \notin Y, a \notin Y \end{cases}$$

By definition, each of these sets is open in Y. But sets of the second and third types are not open in the larger space $\mathbb{R}$.

Note: These sets form a basis for the order topology on Y. Thus, we see that in the case of the set $Y = [0,1]$, its subspace topology and its order topology are the same.

Example 1.7. Let Y be the subset $[0,1) \cup \{2\} \in \mathbb{R}$. In the subspace topology on Y the one-point set $\{2\}$ is open, because $\{2\} = (\frac{3}{2}, \frac{5}{2}) \cap Y$. But in the order topology on Y, the set $\{2\}$ is not open. Any basis element for the order topology on Y that contains 2 is of the form

$$\{x | x \in Y\, and\, a < x \leq 2\}$$

for some $a \in Y$, such a set necessarily contains points of Y less than 2.

Example 1.8. Let $I = [0,1]$. The dictionary order on $I \times I$ is just the restriction to $I \times I$ of the dictionary order on the plane $\mathbb{R} \times \mathbb{R}$. However, the dictionary order topology on $I \times I$ is not the same as the subspace topology on $I \times I$ obtained from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$.

For example, the set $\{l/2\} \times (1/2, 1]$ is open in $I \times I$ in the subspace topology, but not in the order topology. See Figure 1.

Definition 1.9. The set $I \times I$ in the dictionary order topology is called the **ordered square**, and is denoted by I_0^2.

The anomaly seen in Examples 2 and 3 does not occur for intervals or rays in an ordered set X.

Definition 1.10. Given an ordered set X, let us say that a subset Y of X is convex in X if for each pair of points $a < b$ of Y, the entire interval (a,b) of points of X lies in Y.

Note: Intervals and rays in X are convex in X.

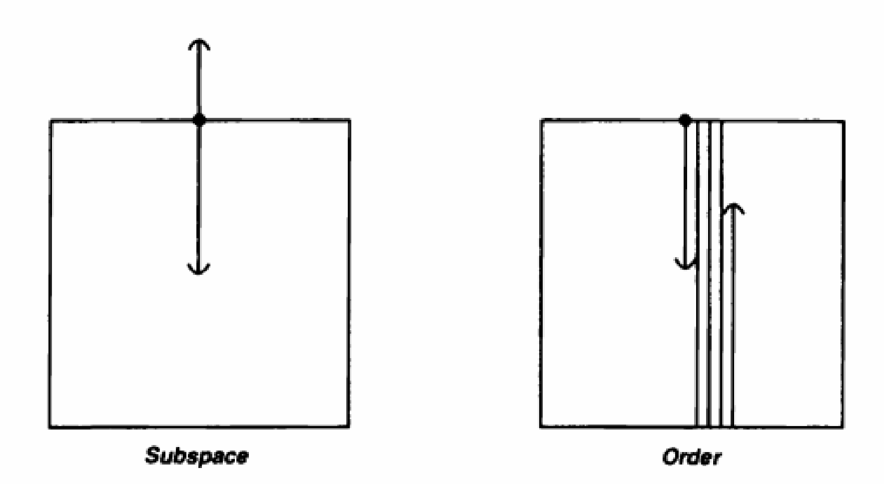

Figure 1: Subspace and order

Theorem 1.11. *Let X be an ordered set in the order topology, let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Proof. Consider the ray $(a, +\infty)$ in X. If $a \in Y$, then

$$(a, +\infty) \cap Y = \begin{cases} \{x : x \in Y, x > a\} & \text{a} \in \text{Y} \\ Y & \text{a=lower bound on Y, a} \notin \text{Y} \\ \phi & \text{a=upper bound on Y, a} \notin \text{Y} \end{cases}$$

A similar remark shows that the intersection of the ray (∞, a) with Y is either an open ray of Y, or Y itself, or empty. Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a \cap Y)$ form a sub-basis for the subspace topology on Y, and since each is open in the order topology, the order topology contains the subspace topology.
To prove the reverse, note that any open ray of Y equals the intersection of an open ray of X with Y, so it is open in the subspace topology on Y. Since the open rays of Y are a sub-basis for the order topology on Y, this topology is contained in the subspace topology. □

2 Closed Sets

Some of the basic concepts associated with topological spaces such as closed set, closure of a set and limit point will be discussed.

Definition 2.1 (Closed set). A subset A of a topological space X is said to be closed if the set $X - A$ is open.

Example 2.2. The closed interval $[a, b] \subseteq \mathbb{R}$ is closed.because its complement $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, +\infty)$ is open.
Similarly, closed rays $[a, +\infty) \subseteq \mathbb{R}$ and $(-\infty, a] \subseteq \mathbb{R}$ are closed.
The subset $[a, b)$ of $\mathbb{R}$ is neither open nor closed.

Example 2.3. In the plane $\mathbb{R}^2$, the set $\{x \times y | x \geq 0 \, and \, y \geq 0\}$ is closed, because its complement is the union of the two sets $(\infty, 0) \times \mathbb{R}$ and $\mathbb{R} \times (-\infty, 0)$, each of which is a product of open sets of $\mathbb{R}$ and is, therefore, open in $\mathbb{R}^2$.

Example 2.4. In the finite complement topology on a set X, the closed sets consist of X itself and all finite subsets of X.

Example 2.5. In the discrete topology on the set X, every set is open, it follows that every set is closed as well.

Example 2.6. Consider the following subset of the real line: $Y = [0, 1] \cup (2, 3)$, in the subspace topology. In this space, the set $[0, 1]$ is open, since it is the intersection of the open set $(1/2, 3/2)$ of $\mathbb{R}$ with Y. Similarly, $(2, 3)$ is open as a subset of Y; it is even open as a subset of $\mathbb{R}$. Since $[0, 1]$ and $(2, 3)$ are complements in Y of each other, we conclude that both $[0, 1]$ and $(2, 3)$ are closed as subsets of Y.

The collection of closed subsets of a space X has properties similar to those satisfied by the collection of open subsets of X:

Theorem 2.7. *Let X be a topological space. Then the following conditions hold:*

1. *ϕ and X are closed.*
2. *Arbitrary intersections of closed sets are closed.*
3. *Finite unions of closed sets are closed.*

Proof. 1. ϕ and X are closed because they are the complements of the open sets X and ϕ, respectively.

2. Given a collection of closed sets we apply De Morgan's law,

$$X \setminus \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X \setminus A_\alpha).$$

Since the sets $X \setminus A_\alpha$ are open by definition, the right side of this equation represents an arbitrary union of open sets, and is thus open. Therefore, $\bigcap A_\alpha$ is closed.

3. Similarly, if A_i is closed for $i \in [n]$, consider the equation

$$X \setminus \bigcup_{\alpha \in J}^{n} A_i = \bigcap_{\alpha \in J}^{n} (X \setminus A_i).$$

The set on the right side of this equation is a finite intersection of open sets and is therefore open. Hence $\cup A_i$ is closed.

$\square$