Lecture 16: The subspace topology, Closed sets

1 Closed Sets and Limit Points

Definition 1.1. A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.

Theorem 1.2. *Let Y be a subspace of X . Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Proof. Assume that $A = C \cap Y$, where C is closed in X . Then $X - C$ is open in X , so that $(X - C) \cap Y$ is open in Y , by definition of the subspace topology. But $(X - C) \cap Y = Y - A$. Hence $Y - A$ is open in Y , so that A is closed in Y . Conversely, assume that A is closed in Y .Then $Y - A$ is open in Y , so that by definition it equals the intersection of an open set U of X with Y . The set $X - U$ is closed in X , and $A = Y \cap (X - U)$, so that A equals the intersection of a closed set of X with Y , as desired. . □

Theorem 1.3. *Let Y be a subspace of X . If A is closed in Y and Y is closed in X , then A is closed in X .*

Definition 1.4. Given a subset A of a topological space X , the **interior** of A is defined as the union of all open sets contained in A, and the **closure** of A is defined as the intersection of all closed sets containing A.

$$IntA = \cup\{U \subseteq A | U \in \mathcal{T}\}$$

$$Closure\ \bar{A} = \cap\{\mathcal{F} \subseteq A | X - \mathcal{F} \in \mathcal{T}\}$$

If A is open, $A = IntA$; while if A is closed, $A = \bar{A}$; furthermore

$$IntA \subseteq A \subseteq \bar{A}$$

Theorem 1.5. *Let Y be a subspace of X ; let A be a subset of Y ; let $\bar{A}$ denote the closure of A in X . Then the closure of A in Y equals $\bar{A} \cap Y$.*

Proof. Let B denote the closure of A in Y . The set $\bar{A}$ is closed in X , so $\bar{A} \cap Y$ is closed in Y by Theorem 1.2. Since $A \cap Y$ contains A, and since by definition B equals the intersection of all closed subsets of Y containing A, we must have $B \subset \bar{A} \cap Y$. On the other hand, we know that B is closed in Y. Hence by Theorem 1.2, $B = C \cap Y$ for some set C closed in X . Then C is a closed set of X containing A; because $\bar{A}$ is the intersection of all such closed sets, we conclude that $\bar{A} \subset C$. Then $(\bar{A} \cap Y) \subset (C \cap Y) = B$. □

Definition 1.6. A **neighborhood** of a point $x \in X$ is an open set U containing x

Theorem 1.7. *Let A be a subset of the topological space X .*

1. *Then $x \in \bar{A}$ if and only if every open set U containing x intersects A .*
2. *Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A .*

Proof. Consider the statement in 1. It is a statement of the form $P \leftrightarrow Q$. Let us transform each implication to its contrapositive, thereby obtaining the logically equivalent statement $(notP) \leftrightarrow (notQ)$. Written out, it is the following:

$$x \notin \bar{A} \longleftrightarrow \textit{there exists an open set } U \textit{ containing } x \textit{ that does not intersect } A$$

the set $U = X - \bar{A}$ is an In this form, our theorem is easy to prove. If x is not in A, open set containing x that does not intersect A, as desired. Conversely, if there exists an open set U containing x which does not intersect A, then $X - U$ is a closed set containing A. By definition of the closure A, x cannot be in A. Statement 2 follows readily. If every open set containing x intersects A, so does every basis element B containing x, because B is an open set. Conversely, if every basis element containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x. □

Example 1.8.

1. Let X be the real line $\mathbb{R}$. If $A = (0, 1]$, then $\bar{A} = [0, 1]$, for every neighborhood of 0 intersects A, while every point outside $[0, 1]$ has a neighborhood disjoint from A
2. If $B = \{1\ n|n \in \mathbb{Z}_+\}$, then $\bar{B} = \{0\} \cup B$.
3. if $C = \{0\} \cup (1, 2),\ \textit{then}\ \bar{C} = \{0\} \cup [1, 2]$.
4. If $\mathbb{Z}_+$ is the set of positive integers, then $\bar{\mathbb{Z}}_+ = \mathbb{Z}_+$

Definition 1.9. If A is a subset of the topological space X and if x is a point of X , we say that x is a **limit point** (or cluster point, or point of accumulation) of A if every neighborhood of x intersects A in some point other than x itself. x is a limit point of A if it belongs to the closure of $A - \{x\}$

Example 1.10.

1. Consider the real line $\mathbb{R}$. If $A = (0, 1]$, then the point 0 is a limit point of A and so is the point 0.5 . In fact, every point of the interval $[0, 1]$ is a limit point of A, but no other point of $\mathbb{R}$ is a limit point of A.
2. If $B = \{1/n | n \in \mathbb{Z}_+\}$, then 0 is the only limit point of B. Every other point x of $\mathbb{R}$ has a neighborhood that either does not intersect B at all, or it intersects B only in the point x itself. If $C = \{0\} \cup (1, 2)$, then the limit points of C are the points of the interval $[1, 2]$.
3. If $\mathbb{R}_+$ is the set of positive reals, then every point of $\{0\} \cup \mathbb{R}_+$ is a limit point of $\mathbb{R}_+$.

Theorem 1.11. *Let A be a subset of the topological space X ; let A' be the set of all limit points of A . Then*

$$\bar{A} = A \cup A'$$

Proof. If x is in A' , every neighborhood of x intersects A (in a point different from x). Therefore, by Theorem 1.7, x belongs to $\bar{A}$. $A' \subset \bar{A}$. Since by definition $A \subset \bar{A}$ it follows that $A \cup A' \subset \bar{A}$. To demonstrate the reverse inclusion, we let x be a point of $\bar{A}$ and show that $x \in A \cup A'$. If x happens to lie in A, it is trivial that $x \in A \cup A'$; suppose that x does not lie in A. since $x \in \bar{A}$. we know that every neighborhood U of x intersects A; because $x \notin A$, the set U must intersect A in a point different from x. Then $x \in A'$,so that $x \in A \cup A'$, as desired. □

2 Housdorff Spaces

Definition 2.1. Let $(X, \mathcal{T})$ be topological space. A sequence $\{x_n : n \in \mathbb{N}\} \subseteq X$ **converges** to $x_0 \in X$ if for all U neighborhood of x_0 $\exists$ $\mathbb{N}$ such that $x_n \in U$ for all $n \geq \mathbb{N}$

Definition 2.2. A topological space X is called a **Hausdorff space** if for each pair x_1 , x_2 of distinct points of X , there exist neighborhoods U_1 and U_2 of x_1 and x_2 , respectively that are disjoint.

Theorem 2.3. *Every finite point set in a Hausdorff space X is closed.*

Proof. It suffices to show that every one-point set $\{x_0\}$ is closed. If x is a point of X different from x_0 , then x and x_0 have disjoint neighborhoods U and V , respectively. Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the set $\{x_0\}$. As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself, so that it is closed. □

Example 2.4. the real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed. The condition that finite point sets be closed has been given a name of its own: it is called the T_1 **axiom**.

Theorem 2.5. *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A .*

Proof. If every neighborhood of x intersects A in infinitely many points, it certainly intersects A in some point other than x itself, so that x is a limit point of A. Conversely, suppose that x is limit point of A, and suppose some neighborhood U of x intersects A in only finitely many points. Then U also intersects $A - \{x\}$ in finitely many points; let $\{x_1, ..., x_m\}$ be the points of $U \cap (A - \{x\})$. The set $X - \{x_1, ..., x_m\}$ is an open set of X , since the finite point set $\{x_1, ..., x_m\}$ is closed; then

$$U \cap (X - \{x_1, ..., x_m\})$$

is a neighborhood of x that intersects the set $A - \{x\}$ not at all. This contradicts the assumption that x is a limit point of A. □

Theorem 2.6. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X .*

Proof. Suppose that x_n is a sequence of points of X that converges to x. If $y = x$, let U and V be disjoint neighborhoods of x and y, respectively. Since U contains x_n for all but finitely many values of n, the set V cannot. Therefore, x_n cannot converge to y. □

Definition 2.7. If the sequence x_n of points of the Hausdorff space X converges to the point x of X , we often write $x_n \to x$, and we say that x is the **limit** of the sequence x_n.