Lecture 17: Continuous Functions

1 Continuous Functions

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

Definition 1.1 (Continuous Function). A function $f : X \to Y$ is said to be **continuous** if the inverse image of every open subset of Y is open in X. In other words, if $V \in \mathcal{T}_Y$, then its inverse image $f^{-1}(V) \in \mathcal{T}_X$.

Proposition 1.2. *A function $f : X \to Y$ is continuous iff for each $x \in X$ and each neighborhood N of $f(x)$ in Y, the set $f^{-1}(N)$ is a neighborhood of x in X.*

Proof. Let x be an arbitrary element of X and N an arbitrary neighborhood of $f(x)$ in Y. Then, $f^{-1}(N)$ and contains x and by definition, is open in X. Hence, for each $x \in X$ and each neighborhood N of $f(x)$ in Y, the set $f^{-1}(N)$ is a neighborhood of x in X. Conversely, let for each $x \in X$ and each neighborhood N of $f(x)$ in Y, the set $f^{-1}(N)$ is a neighborhood of x in X. Let V be an arbitrary open subset of Y.

i) If $V \cap f(X) = \emptyset$, where $f(X)$ is the range of f, then $f^{-1}(V) = \emptyset$ and hence is open in X.

ii) If $V \cap f(X) \neq \emptyset$, then V is a neighborhood of each of its points (let $f(x)$ be one such point for some $x \in X$). By assumption, $f^{-1}(V)$ $(\subseteq X)$ must be a neighborhood of each of its points (including the said x) in X and hence, $f^{-1}(V)$ is open in X.

□

Note 1. Continuity of a function depends not only on f but also on its domain and co-domain topologies X and Y.

Example 1.3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ be a function.

i) If f is a constant map, i.e., $f(x) = y$ for all $x \in X$ and some $y \in Y$, then f is continuous for all topologies on X and Y because for any open subset V of Y, $f^{-1}(V) = \emptyset$ (if $y \notin V$) or X (if $y \in V$), both of which are always open in any topology on X.

ii) If $\mathcal{T}_X = \mathcal{P}(X)$, i.e., $(X, \mathcal{T}_X)$ is the discrete topology, then f is continuous for any topology on Y because for any open subset V of Y, $f^{-1}(V)$ is in $\mathcal{P}(X)$ and hence is open in X.

iii) If $\mathcal{T}_Y = \{\emptyset, Y\}$, i.e., $(Y, \mathcal{T}_Y)$ is the trivial topology, then f is continuous for any topology on X because $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$, both of which are always open in any topology on X.

iv) The identity mapping from $(X, \mathcal{T}_X)$ to $(X, \mathcal{T}_X)$ is continuous because for any $U \in \mathcal{T}_X$ (co-domain topology), $f^{-1}(U) = U \in \mathcal{T}_X$ (domain topology).

Example 1.4.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces defined as follows:

$$X = \{R, G, B\} \qquad \mathcal{T}_X = \{\emptyset, \{R\}, \{B\}, \{R, G\}, \{R, B\}, X\}$$

$$Y = \{1, 2, 3\} \qquad \mathcal{T}_Y = \{\emptyset, \{1\}, \{1, 2\}, Y\}$$

Let f and g be bijective mapping defined as $f(R) = 1$, $f(G) = 2$ and $f(B) = 3$. Then, f is continuous since

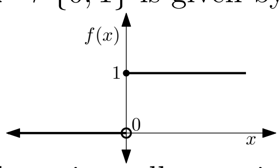
$$f^{-1}(\emptyset) = \emptyset,\ f^{-1}(\{1\}) = \{R\}, f^{-1}(\{1, 2\}) = \{R, G\},\ f^{-1}(Y) = X$$

all of which are open in X. However, its inverse map g, with $g(1) = R$, $g(2) = G$ and $g(3) = B$, is not continuous since

$$g^{-1}(\{B\}) = \{3\} \notin \mathcal{T}_Y \text{ and } g^{-1}(\{R, B\}) = \{1, 3\} \notin \mathcal{T}_Y.$$

Example 1.5. The unit step function $u : \mathbb{R} \to \{0, 1\}$ is given by

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0. \end{cases}$$



Let $\mathbb{R}$ be equipped with the standard topology, i.e., all open intervals are open, and the set $\{0, 1\}$ be equipped with the discrete topology. Then, $u^{-1}(0) = (-\infty, 0)$ is open in the standard topology on $\mathbb{R}$, but $u^{-1}(1) = [0, \infty)$ is not. Hence, the unit step function is discontinuous.

Example 1.6. Let $\mathbb{R}$ and $\mathbb{R}_l$ denote the set of real numbers equipped with the standard and lower limit topology respectively, and $f : \mathbb{R} \to \mathbb{R}_l$ and $g : \mathbb{R}_l \to \mathbb{R}$ be identity functions, i.e., $f(x) = g(x) = x$, for every real number x. Then, f is not continuous because the inverse image of the open set $[a, b)$ in $\mathbb{R}_l$ is $[a, b)$ which is not open in the standard topology. But g is continuous because the inverse image of open interval (a, b) in the standard topology on $\mathbb{R}$ is open in $\mathbb{R}_l$ ($g^{-1}((a, b)) = (a, b) = \cup_{n\in\mathbb{N}}[a + {}^1/_n, b)$ and countable union of open sets is open).

Example 1.7. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at $x_0 \in \mathbb{R}$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } |x - x_0| < \delta \ \to \ |f(x) - f(x_0)| < \epsilon,$$

where both the domain and co-domain topologies are the standard topology on $\mathbb{R}$. The equivalence of this definition of continuity to the open-set definition of continuity at x_0 is shown below.

Let f be continuous at x_0 by the open set definition, i.e., inverse image of every open set containing x_0 is open. Given any $\epsilon > 0$, the interval $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is open in the co-domain topology and hence, $f^{-1}(V)$ is open in the domain topology. Since $f^{-1}(V)$ contains x_0, it contains a basis (a, b) about x_0 (since for every open set S and every $s \in S$, there exists a basis B_s such that $s \in B_s \subseteq S$). Let δ be minimum of $x_0 - a$ and $b - x_0$. Then if $|x - x_0| < \delta$, x must be in (a, b) and $f^{-1}(V)$ (since $(a, b) \subseteq f^{-1}(V)$). Hence $f(x) \in V$ and $|f(x) - f(x_0)| < \epsilon$ as required.

Now, let f be ϵ−δ−continuous at $x \in \mathbb{R}$ and V be an open set in the co-domain topology containing $f(x)$. Since V is open and $f(x) \in V$, there exists some $\epsilon > 0$, such that $(f(x) - \epsilon, f(x) + \epsilon) \subseteq V$. By continuity at x, there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq f^{-1}(V)$. Since $(x - \delta, x + \delta)$ is open in the domain topology and the choices of ϵ and V were arbitrary, inverse image of every open set containing x is open as required by the open set definition of continuity at x. Note that if an open set V in co-domain topology does not intersect the range of f, then $f^{-1}(V) = \emptyset$, which is open in the domain topology.

Following are some properties of continuity.

1. For two topologies $\mathcal{T}_X$ and $\mathcal{T}'_X$ on X, the identity map 1_X from $(X, \mathcal{T}_X)$ to $(X, \mathcal{T}'_X)$ is continuous iff $\mathcal{T}_X$ is finer than $\mathcal{T}'_X$.

   Proof. Let $f = 1_X$. Since the map is identity, $f^{-1}(S) = S$ for any subset S of X. Let the identity map be continuous. Then, for any V in $\mathcal{T}'_X$, $f^{-1}(V)$ is in $\mathcal{T}_X$. Since $f^{-1}(V) = V$, this means that V is also in $\mathcal{T}_X$. Thus, $\mathcal{T}'_X \subseteq \mathcal{T}_X$, i.e., $(X, \mathcal{T}_X)$ is finer than $(X, \mathcal{T}'_X)$. Conversely, let $\mathcal{T}_X$ is finer than $\mathcal{T}'_X$. Then, any set S in $\mathcal{T}'_X$ is also in $\mathcal{T}_X$. For any V in $\mathcal{T}'_X$, $f^{-1}(V)$ is in $\mathcal{T}_X$ because $f^{-1}(V) = V$ and V is in $\mathcal{T}_X$. Thus, the identity map is continuous. □

2. A continuous map remains continuous if the domain topology becomes finer or the co-domain topology becomes coarser.

   Proof. Let $(X, \mathcal{T}_1)$, $(X, \mathcal{T}_2)$, $(Y, \mathcal{S}_1)$ and $(Y, \mathcal{S}_2)$ be topologies with $\mathcal{T}_1$ and $\mathcal{S}_1$ finer than $\mathcal{T}_2$ and $\mathcal{S}_2$ respectively. Let f be a continuous map from $(X, \mathcal{T}_2)$ to $(Y, \mathcal{S}_1)$.

   i) Let V be in $\mathcal{S}_1$. Then, $f^{-1}(V)$ is in $\mathcal{T}_2$, since f is continuous, and in $\mathcal{T}_1$, since it is finer than $\mathcal{T}_2$. Thus, f is also a continuous map from $(X, \mathcal{T}_1)$ to $(Y, \mathcal{S}_1)$.

   ii) Let V be in $\mathcal{S}_2$. Since, $\mathcal{S}_1$ is finer than $\mathcal{S}_2$, it contains V. Also $\mathcal{T}_2$ contains $f^{-1}(V)$ since f is a continuous. Thus, f is also a continuous map from $(X, \mathcal{T}_2)$ to $(Y, \mathcal{S}_2)$.

   □

Note 2. i) From Property 1, it can be inferred that, continuity of a bijective function $f : X \to Y$ does not guarantees continuity of its inverse (cf. Examples 1.4 and 1.6).

ii) In Example 1.6, had f been the identity map from $\mathbb{R}$ to itself then it would have been continuous but replacing the co-domain topology with a finer topology ($\mathbb{R}_l$) renders it discontinuous.

To test the continuity of a map from a topological space on X to that on Y, checking whether inverse image of each open set in Y is open in X is not necessary.

Theorem 1.8. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ be a function. Then, the following statements are equivalent:*

1. *f is continuous.*

2. *Inverse image of every basis element of $\mathcal{T}_Y$ is open.*

3. *Inverse image of every subbasis element of $\mathcal{T}_Y$ is open.*

Thus, to test the continuity of a function it suffices to check openness of inverse images of elements of only a subset of $\mathcal{T}_Y$, viz., its subbasis.

Proof. **(1)→(2)** Let f be continuous. Since every basis element of $\mathcal{T}_Y$ is open, its inverse image will be open.

(2)→(1) Let $\mathcal{B}_Y$ be a basis for $\mathcal{T}_Y$ and let the inverse image of every basis element $B \in \mathcal{B}_Y$ be open in X, i.e., $f^{-1}(B) \in \mathcal{T}_X$. Note that any open set V in Y can be written as a union of the basis elements, i.e., $V = \cup_{j\in J} B_j$, $f^{-1}(V) = \cup_{j\in J} f^{-1}(B_j)$, for some $\{B_1, \ldots, B_{|J|}\} \subseteq \mathcal{B}_Y$. Since union of opens sets is open, $f^{-1}(V)$ is open.

(2)→(3) Since every subbasis element is in the basis it generates, inverse image of every subbasis element of Y is open in X.

(3)→(2) Let $\mathcal{S}_Y$ be subbasis of Y which generates the basis $\mathcal{B}_Y$. Let the inverse image of every subbasis element $S \in \mathcal{S}_Y$ be open in X, i.e., $f^{-1}(S) \in \mathcal{T}_X$. Since any basis element can be written as a finite intersection of subbasis elements, i.e., $B = \cap_{i=1}^{n} S_i$, $f^{-1}(B) = \cap_{i=1}^{n} f^{-1}(S_i)$. Since finite intersection of open sets is open, $f^{-1}(B)$ is open in X.

□

Theorem 1.9. *Let f be a map from a topological space on X to a topological space on Y. Then, the following statements are equivalent:*

1. *f is continuous.*
2. *Inverse image of every closed set of Y is closed in X.*
3. *For each $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*
4. *For every subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$.*
5. *For every subset B of Y, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.*

Proof. **(1)→(2)** Let a subset C of Y be closed. Then, its complement $Y\backslash C$ is open and the inverse image of the complement $f^{-1}(Y\backslash C) = f^{-1}(Y)\backslash f^{-1}(C) = X\backslash f^{-1}(C)$ is open in X. Hence, $f^{-1}(C)$ is closed in X.

(2)→(1) Let V be open in Y. Then, its complement $Y\backslash V$ is closed and the inverse image of the complement $f^{-1}(Y\backslash V) = f^{-1}(Y)\backslash f^{-1}(V) = X\backslash f^{-1}(V)$ is closed in X. Hence, $f^{-1}(V)$ is open in X.

(1)→(3) Since $f^{-1}(V)$ is an open neighborhood of x, choose $U = f^{-1}(V)$.

(3)→(4) Let $A \subseteq X$ and $x \in \overline{A}$. Let V be a neighborhood $f(x)$ and U be a neighborhood of x such that $f(U) \subseteq V$. Since $x \in \overline{A}$, $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$ (cf. Lecture 5, Theorem 2.3(vii)). Since the choice of V neighborhood of $f(x)$ was arbitrary, every neighborhood of $f(x)$ intersects $f(A)$. Hence, $f(x) \in \overline{f(A)}$ and $f(\overline{A}) \subseteq \overline{f(A)}$.

(4)→(5) Let $A = f^{-1}(B)$. Then, by (4), $f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(B))} = \overline{B}$. Hence, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

(5)→(2) Let $B \subseteq Y$ be closed; then, $\overline{B} = B$ since a set is closed iff it is equal to its closure. Then, by (5), $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$ and since $f^{-1}(B) \subseteq \overline{f^{-1}(B)}$ is always true, $\overline{f^{-1}(B)} = f^{-1}(B)$. Hence, $f^{-1}(B)$ is closed (being equal to its closure).

□

2 Homeomorphism

Definition 2.1 (Homeomorphism). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ be a bijection. If both f and its inverse $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

The two spaces are said to be *homeomorphic* and each is a *homeomorph* of the other. If a map is a homeomorphism, then so is its inverse. Composition of any two homeomorphisms is again a homeomorphism.

The requirement the f^{-1} be continuous means that for any U open in X, its inverse image under f^{-1} be open in Y. But since the inverse image of U under f^{-1} is same as the image of U under f (cf. Lecture 5, Remark 2(vi)), another way to define a homeomorphism is to say that it is a bijective map $f : X \to Y$ such that $f(U)$ is open iff U is open. Thus, a homeomorphism is a bijection between $\mathcal{T}_X$ and $\mathcal{T}_Y$. Consequently, any property of X expressed in terms of $\mathcal{T}_X$ (or the open sets), yields, via f, the corresponding property for Y. Such a property is called a *topological property* of X.

Let $f : X \to Y$ be an injective continuous map and $Z = f(X) \subset Y$ be its range, considered as a subspace of Y. Then, the map obtained by restricting Y to Z, $f' : X \to Z$ is a bijection. If f' happens to be a homeomorphism, then we say that $f : X \to Y$ is a *topological imbedding*, or simply an *imbedding*, of X in Y.

Example 2.2. Let $\mathbb{R}$ be equipped with the trivial, standard or discrete topology. For every pair of real numbers m and c, the function $f_{m,c} : \mathbb{R} \to \mathbb{R}$ defined by $f_{m,c}(x) = mx + c, \forall x \in \mathbb{R}$ is a homeomorphism.

Example 2.3. i) The identity map from a topological space to itself is a homeomorphism (Example 1.3(iv)).

ii) The map f in Examples 1.4 and the map g in1.6 are both continuous and bijective but not homeomorphic because their inverse maps are not continuous.

Example 2.4. i) Two discrete spaces are homeomorphic iff there is a bijection between them, i.e., iff they have the same cardinality. This is true because every function on a discrete space is continuous, no matter the co-domain topology (Example 1.3(ii)).

ii) Two trivial topologies are homeomorphic iff there is a bijection between them. This holds because every function to a trivial topology is continuous regardless of the domain topology (Example 1.3(iii)).

Proposition 2.5. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ be a function. Then, the following statements are equivalent:*

i) f is a homeomorphism.

ii) U is open in X iff $f(U)$ is open in Y.

iii) C is closed in X iff $f(C)$ is closed in Y.

iv) V is open in Y iff $f^{-1}(V)$ is open in X.

v) D is closed in Y iff $f^{-1}(D)$ is closed in X.

3 Constructing Continuous Functions

Some rules for constructing continuous functions are given below.

Theorem 3.1. *Let X, Y and Z be topological spaces.*

1. *(Constant function) If $f : X \to Y$ defined as $f(x) = y$ for all $x \in X$ and some $y \in Y$, then f is continuous.*
2. *(Inclusion) If A is a subspace of X, then the inclusion function $j : A \to X$ is continuous. ($j(a) = a, \forall\, a \in A$)*
3. *(Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is their composition $g \circ f : X \to Z$.*
4. *(Restricting the domain) If $f : X \to Y$ is continuous and A is a subspace of X, then the restriction of f to A, $f|A : A \to Y$ is also continuous.*
5. *(Restricting or expanding the range) Let $f : X \to Y$ be continuous.*
    a) *If Z is subspace of Y containing the range $f(X)$, then the function $g : X \to Z$ obtained by restricting the co-domain topology is continuous.*

b) If Z is space containing Y as a subspace, then the function $h : X \to Z$ obtained by expanding the co-domain topology is continuous.

6. *(Local formulation of continuity)The map $f : X \to Y$ is continuous if X cna be written as the union of open sets U_α such that $f|U_\alpha$ is continuous for each α.*

Proof. i) See Example 1.3(i).

ii) If U is open in X, then $j^{-1}(U) = U \cap A$ is open in A by definition of subspace topology.

iii) If W is open in Z, then $g^{-1}(W)$ is open in Y since g is continuous. Since f is continuous, $f^{-1}(g^{-1}(W))$ is open in X. Thus, $g \circ f$ is continuous (since $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$).

iv) $f|A = j \circ f$, both of which are continuous and composition of continuous maps is continuous.

v) (a) Let W be open in Z. Then, $B = Z \cap U$ for some U open in Y. Since $f(Z) \subseteq Z$, $f^{-1}(B) = f^{-1}(U)$ and is open in X because $f^{-1}(U)$ is open in X. (b) Let $j : Y \to Z$ be the inclusion map. Then, $h = f \circ j$.

vi) Let V be open in Y. Then,

$$f^{-1}(V) \cap U_\alpha = (f|U_\alpha)^{-1}(V)$$

and is open in U_α and hence open in X. But

$$f^{-1}(V) = \bigcup_\alpha \left(f^{-1}(V) \cap U_\alpha\right),$$

so that V is also open in X.

□

Theorem 3.2 (The Pasting Lemma). *Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ e continuous maps. If $f(x) = g(x)$ for every $x \in A \cap B$, the f and g combine to give a continuous map $h : X \to Y$, defined as*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

Proof. Let C be a closed subset of Y. Then, $h{-}1(C) = f^{-1}(C) \cup g^{-1}(C)$ and is closed in X since each of $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X. □

The pasting lemma hold even if A and B are open in X and is a special case of Theorem 3.1(vi).

Theorem 3.3 (Maps into Products). *Let $f : A \to X \times Y$ be defined as $f(a) = (f_1(a), f_2(a))$. Then f is continuous iff both the co-ordinate functions $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.*

Proof. The projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ onto the first and second factor space are continuous since $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ are open if U and V are open in X and Y respectively. Note that $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$. If f is continuous, then so are f_1 and f_2 (composites of continuous functions). Conversely, let f_1 and f_2 are continuous. Let $U \times V$ be a basis element of $X \times Y$. A point a is in $f^{-1}(U \times V)$ iff $f(a) \in U \times V$, i.e., iff $f_1(a) \in U$ and $f_2(a) \in V$. Hence, $f^{-1}(U \times V) = f^{-1}(U) \cap f^{-1}(V)$ and is open in A since both $f^{-1}(U)$ and $f^{-1}(V)$ are open. Thus, since inverse image of every basis element is open, f is continuous (by Theorem 1.8(2)) □