Lecture 20: Compactness

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1 Compact spaces

Definition 1.1. A collection $\mathcal{A}$ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

Definition 1.2. A space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite sub-collection that also covers X.

Example 1.3. The real line $\mathbb{R}$ is not compact, for the covering of $\mathbb{R}$ by open intervals $A = \{(n, n+2) | n \in \mathbb{Z}\}$ contains no finite sub-collection that covers $\mathbb{R}$.

Example 1.4. The following subspace of $\mathbb{R}$ is compact:$X = \{0\} \cup \{1/n | n \in \mathbb{Z}_+\}$ Given an open covering $\mathcal{A}$ of X, there is an element U of $\mathcal{A}$ containing 0. The set U contains all but finitely many of the points $1/n$; choose, for each point of X not in U, an element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$, along with the element U, is a finite sub-collection of $\mathcal{A}$ that covers X.

Example 1.5. Any space X containing only finitely many points is necessarily compact, because in this case every open covering of X is finite.

Example 1.6. The interval $(0, 1]$ is not compact; the open covering $\mathcal{A} = \{(1/n, 1] | n \in \mathbb{Z}_+\}$ contains no finite sub-collection covering $(0, 1]$. The interval $(0, 1)$ is also not compact by the same argument applies. On the other hand, the interval $[0, 1]$ is compact.

We shall prove some general theorems that show us how to construct new compact spaces out of existing ones.

Lemma 1.7. *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite sub-collection covering Y.*

Proof. Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X. Then the collection $\{A_\alpha \cap Y | \alpha \in J\}$ is a covering of Y by sets open in Y;

hence a finite sub-collection $\{A_{\alpha_1} \cap Y, ..., A_{\alpha_n} \cap Y\}$ covers Y. Then $\{A_{\alpha_1}, ..., A_{\alpha_n}\}$ is a sub-collection of $\mathcal{A}$ that covers Y.

Conversely, suppose the given condition holds; we wish to prove Y is compact. Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α, choose a set A_α open in X such that $A'_\alpha = A_\alpha \cap Y$. The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X. By hypothesis, some finite sub-collection $\{A_{\alpha_1}, ..., A_{\alpha_n}\}$ covers Y. Then $\{A'_{\alpha_1}, ..., A'_{\alpha_n}\}$ is a sub-collection of $\mathcal{A}'$ that covers Y. □

Theorem 1.8. *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets open in X, let us form an open covering $\mathcal{B}$ of X by adjoining to $\mathcal{A}$ the single open set $X - Y$, that is, $\mathcal{B} = \mathcal{A} \cup \{X - Y\}$. Some finite sub-collection of $\mathcal{B}$ covers X. If this sub-collection contains the set $X - Y$, discard $X - Y$; otherwise, leave the sub-collection alone. The resulting collection is a finite sub-collection of $\mathcal{A}$ that covers Y. □

Theorem 1.9. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of the Hausdorff space X. We shall prove that $X - Y$ is open, so that Y is closed.
Let x_0 be a point of $X - Y$. We show there is a neighborhood of x_0 that is disjoint from Y. For each point y of Y, let us choose disjoint neighborhoods and of the points x_0 and y, respectively (using the Hausdorff condition). The collection $\{V_y | y \in Y\}$ is a covering of Y by sets open in X; therefore, finitely many of them $V_{y_1}, ..., V_{y_n}$ cover Y. The open set $V = V_{y_1} \cup ... \cup V_{y_n}$ contains Y, and it is disjoint from the open set $U = U_{y_1} \cap ... \cap U_{y_n}$ formed by taking the intersection of the corresponding neighborhoods of x_0. For if z is a point of V, then $z \in V_{y_i}$ for some i, hence $z \notin U_y$, and so $z \notin U$. Then U is a neighborhood of x_0 disjoint from Y, as desired. □

Lemma 1.10. *If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.*

Example 1.11. Once we prove that the interval $[a, b]$ in $\mathbb{R}$ is compact, it follows from Theorem 1.8 that any closed subspace of $[a, b]$ is compact. On the other hand, it follows from Theorem 1.9 that the intervals $(a, b]$ and (a, b) in $\mathbb{R}$ cannot be compact because they are not closed in the Hausdorff space $\mathbb{R}$.

Example 1.12. Consider the finite complement topology on the real line. The only proper subsets of $\mathbb{R}$ that are closed in this topology are the finite sets. But every subset of $\mathbb{R}$ is compact in this topology, as can be checked.

Theorem 1.13. *The image of a compact space under a continuous map is compact.*

Proof. Let $f : X \to Y$ be continuous; let X be compact. Let A be a covering of the set $f(X)$ by sets open in Y. The collection $\{f^{-1}|A \in \mathcal{A}\}$ is a collection of sets covering X; these sets are open in X because f is continuous. Hence finitely many of them, say $f^{-1}(A_1), ..., f^{-1}(A_n)$, cover X. Then, the sets $A_1, ..., A_n$ cover $f(X)$. □

Theorem 1.14. *Let $I : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We shall prove that images of closed sets of X under f are closed in Y; this will prove continuity of the map f^{-1}. If A is closed in X, then A is compact, by Theorem 1.8. Therefore, by the theorem just proved, $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed in Y, by Theorem 1.9. □

Theorem 1.15. *The product of finitely many compact spaces is compact.*

Lemma 1.16 (The tube lemma). *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of in X.*

Proof. Let us cover $x_0 \times Y$ by basis elements $U \times V$ lying in N. The space $x_0 \times Y$ is compact being homeomorphic to Y, we can find finite sub-cover for $x_0 \times Y$ as $U_1 \times V_1, ..., U_n \times V_n$. We can assume $U_i \times V_i$ intersects $x_0 \times Y$, $W := \cap_{(i=1)}^{n} U_i$ is neighborhood of x_0. The chosen $\{U_i \times V_i\}$ cover $W \times Y$ and lie in N. □

Theorem 1.17. *The product of finitely many compact spaces is compact.*

Proof. Let X and Y be compact spaces. Let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, $x_0 \times Y$ is compact and may be covered by finitely many elements of $\mathcal{A}$, say, $A_1, ..., A_m$. Their union $N = \cup_{i=1}^{n} A_i$ is an open set containing $x_0 \times Y$. We can find a tube $W \times Y$ about $x_0 \times Y$ where W is open in X. Then, $W \times Y$ is covered by finitely many elements $A_1, ..., A_m$. For each $x \in X$ find neighborhood of x such that tube $W_x \times Y$ is covered by finitely many elements of $\mathcal{A}$. Now $\{W_x : x \in X\}$ is an open covering of X, hence, there exists finite sub-cover $\{W_1, ..., W_k\}$ that covers X. Union of the tubes $W_1 \times Y, ..., W_k \times Y$ covers $X \times Y$. Similar argument can be used to complete the proof by induction. □