Lecture 21 : Measurable Spaces

1 Measurable Spaces

Corollary 1.1. *Let $(X, \mathcal{F}_c)$ be measurable space and $(Y, \mathcal{F}_y)$ be a topological space and $\mathcal{G}$ be a Borel σ-algebra on Y. Let $g : Y \longrightarrow \mathbb{R}$ be a continous function. Then the map $g \circ f : X \longrightarrow \mathbb{R}$ is measurable for each measurable function $f : X \longrightarrow Y$.*

Definition 1.2. Let $\{f_i : i \in I\}$ be a family of maps from a set X into a measurable space (Y, G). The σ algebra generated by $\{f_i : i \in I\}$ denoted $\sigma(\{f_i : i \in I\})$ is intersection of all σ algebra on X that make each f_i for $i \in I$ measurable.

Definition 1.3. Let $\{X_i : i \in I\}$ be an indexed collection of non empty sets. Let $X = \prod_{i \in I} X_i$ and $\pi_i : X \longrightarrow X_i$ be the coordinate maps. If $\mathcal{F}_i$ is a σ algebra on X_i for each i. The product σ algebra is the σ algebra generated by

$$A = \{\pi_{i^-1}(E_i) : E_i \in \mathcal{F}_i, i \in I\}$$

we denote this σ algebra by $\bigotimes_{i \in I} \mathcal{F}_i$. For $I = \{1, 2, ...n\}$ we write

$$\bigotimes_{i=1}^{n} \mathcal{F}_i = \mathcal{F}_1 \bigotimes \mathcal{F}_2 \bigotimes ...... \bigotimes \mathcal{F}_n$$

Proposition 1.4. *If I is countable, then $\bigotimes_{i \in I} \mathcal{F}_i$ is σ algebra generated by*

$$B = \{\prod_{i \in I} E_i : E_i \in \mathcal{F}_i\}$$

Proof. If $E_i \in \mathcal{F}_i$ then $\pi_i^{-1}(E_i) = \prod_{j \in I} E_j$ where

$$E_j = \left\{ \begin{array}{cc} E_i & j = i \\ X_j & j \neq i1 \end{array} \right\}$$

Then

$$\bigcap_{i \in I} \pi_i^{-1}(E_i) \in \sigma(A)$$

Hence $B \subseteq \sigma(A)$. On the other hand, $\pi_i^{-1}(E_i) = \pi_{j \in I} E_j \in B$. Hence, $A \subseteq \sigma(B)$ ☐

Proposition 1.5. *Let $\mathcal{F}_i$ be generated by $A_i, i \in I$. Then $\bigotimes_{i\in I} \mathcal{F}_i$ is generated by*

$$B_1 = \{\pi_i^{-1}(E_i) : E_i \in A_i, i \in I\}$$

If I is countable and $X_i \in E_i$ for all $i \in I$. Then $\bigotimes_{i\in I} \mathcal{F}_i$ is generated by

$$B_2 = \{\pi_{i\in I} E_i : E_i \in A_i\}$$

Proof. Clearly, $B_1 \subseteq A$ Then $\sigma(B_1) \subseteq \bigotimes_{i\in I} \mathcal{F}_i$. On the other hand for each $i \in I$, the collection

$$C_i = \{E \subset X_i : \pi_i^{-1}(E) \in \sigma(B_1)\}$$

is σ algebra on X that contains A_i, Hence $\mathcal{F}_i \subseteq C_i$. In other words, $\pi_i^{-1}(E) \in \sigma(B_1)$ for all $E \in \mathcal{F}_i, i \in I$. Hence $\bigotimes_{i\in I} \mathcal{F}_i \subseteq \sigma(B_1)$

□