Lecture 22: Introduction to Measure Theory

1 Introduction

The measure theory is a natural extension of the concept of measure in Euclidean geometry, i.e., area and volume. To understand the concept of measurable sets and measurability in more general terms we consider the concept of $\sigma-$algebra.

Let X be a non-empty set.

Definition 1.1 (Algebra). An algebra of sets of X is a non-empty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements. In other words, $\mathcal{A} \subseteq \mathcal{P}(X)$ s.t. $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$, and $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$.

Definition 1.2 (σ-algebra). A σ-algebra is an algebra that is closed under countable unions. That is, $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Next, we define two more structures, i..e, $\pi-$system and $\lambda-$system and show their relation with algebra and $\sigma-$ algebra. Consider following set of assumptions

A1 $\phi \in \mathcal{A}$

A2 $X \in \mathcal{A}$

A3 **[Close under complements]** $A \in \mathcal{A} \implies A^c \in \mathcal{A}$.

A4 **[Close under complements]** $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

A5 **[Close under complements]** $A, B \in \mathcal{A}, A \subseteq B \implies B \backslash A \in \mathcal{A}$.

A6 **[Close under finite intersections]** $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$.

A7 **[Close under countable unions]** $A_n \in \mathcal{A}$ for all $n \in \mathbb{N} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

A8 **[Close under increasing limits]** $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}, A_n \nearrow A \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

A9 **[Close under countable union of pairwise disjoint sets]** $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, pairwise disjoint $\implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Next we define various structures in terms of these assumptions.

Definition 1.3. A family $\mathcal{A}$ of subsets of a non-empty X is called an

1. Algebra if it satisfies $A1, A3,$ and $A4$
2. $\sigma-$algebra if it satisfies $A1, A3,$ and $A7$
3. $\pi-$system if it satisfies $A6$
4. $\lambda-$system if it satisfies $A2, A5$ and $A8$

Note that, $A3$ and $A4$ implies $A1, A2, A5,$ and $A6$. Next, we study the properties of the above described structures.

1. Every $\sigma-$algebra is an algebra.
2. Each algebra is a $\pi-$system, and each $\sigma-$algebra is an algebra and a λ-system.
3. A family of sets is a $\sigma-$algebra iff $A1, A3, A6,$ and $A9$ holds.

   Proof. The proof in forward direction follows from the fact that $A3$ and $A7$ together implies $A1, A2, A4, A5, A6, A8$ and $A9$. Next, we present the proof in the reverse direction. Let $B_1 = A_1, B_2 = A_2 \backslash A_1$, i.e., $B_n = A_n \backslash \cup_{j=1}^{n-1} A_j$ and $\cup_{j=1}^{n} B_j = \cup_{j=1}^{n} A_j$. Now, $\{B_n : n \in \mathbb{N}\}$ pairwise disjoint. Hence, $\cup_{n \in \mathbb{N}} A_j = \cup_{n \in \mathbb{N}} B_n \in \mathcal{A}$. □

4. A $\lambda-$system is also a $\pi-$system is also a $\sigma-$algebra.

   Proof. Since $A2$ and $A5$ implies $A3$, i.e., let $A \in \mathcal{A}$, then $X \backslash A \in \mathcal{A}$ (by $A2$ and $A5$). The proof follows by observing that $A3$ and $A6$ together implies $A4$ which is proved next. Let $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. Hence, $B_m = \cup_{n=1}^{m} A_n \in \mathcal{A}$. Since, $B_m \nearrow \cup_{n \in \mathbb{N}} A_n$ then $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ (by $A8$). □

5. There are $\pi-$systems which are not algebras.

   Example 1.4. Let $X = \{1, 2, 3\}$ and $\mathcal{A}_c = \{\{1, 2\}, \{2, 3\}, \{3\}\}$.

6. There are algebras that are not $\sigma-$algebras.

   Proof. Let $X = \mathbb{N}, \mathcal{A} = \{A \subseteq X : A or X \backslash A finite\}$. Let $A \in \mathcal{A}$, the A is finite then $X \backslash A \in \mathcal{A}$ or $X \backslash A$ finite, then $X \backslash A \in \mathcal{A}$. Finite union of finite sets is finite, i.e., $\cup_{i=1}^{n} A_i \in \mathcal{A}$. Finite intersection of finite sets is finite $\cap_{i=1}^{n} A_i \in \mathcal{A}$. A_1 is finite, $X \backslash A_2$ finite, then $X \backslash (A_1 \cup A_2) = (X \backslash A_1) \cap (X \backslash A_2)$. $A_n = \{2n\}$ □

7. There are $\lambda-$systems which are not $\pi-$system.

   Proof. Left as a exercise. □

Note: It is trivial to verify that intersection of any family of $\sigma-$algebra is also $\sigma-$algebra.

Definition 1.5. For a family $\mathcal{A}$ of subsets of a non-empty set X, the intersections of all $\sigma-$algebras on X that contain $\mathcal{A}$ is denoted by $\sigma\mathcal{A}$ and is called the $\sigma-$algebra generated by $\mathcal{A}$.

Example 1.6.

1. $\mathcal{P}(X)$ is a $\sigma-$algebra.
2. $\{\Phi, X\}$ is a $\sigma-$algebra.