Lecture 23: Measures

1 Measures

Definition 1.1. Let $(X, \mathcal{F})$ be a measurable space. A set mapping $\mu : \mathcal{F} \to [0, \infty]$ is called a **measure** if

i. $\mu(\emptyset) = 0$,

ii. **Countable additivity.** $\mu(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$ for all sequences $\{A_n : n \in \mathbb{N}\}$ of pairwise disjoint sets in $\mathcal{F}$.

Definition 1.2. Let X be a non-empty set, equipped with a σ-algebra $\mathcal{F}$, and μ is a measure on $\mathcal{F}$, then $(X, \mathcal{F}, \mu)$ is called a **measure space**.

Theorem 1.3 (Properties of Measures). *Let $(X, \mathcal{F}, \mu)$ be a measure space, and $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{F}$, then the following are true.*

1. ***Finite additivity.*** *If $\{A_i : i \in [n]\}$ a finite collection of pairwise disjoint sets, then $\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$.*

2. ***Monotonicity.*** *If $A_i \subseteq A_j$, then $\mu(A_i) = \mu(A_j)$.*

3. ***Continuity from below.*** *If $\{A_n : n \in \mathbb{N}\}$ is an increasing sequence of sets, i.e. $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then*

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\in\mathbb{N}} \mu(A_n) = \sup_{n\in\mathbb{N}} \mu(A_n).$$

4. ***Continuity from below.*** *If $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of sets, i.e. $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$ and $\mu(A_1) < \infty$, then*

$$\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\in\mathbb{N}} \mu(A_n) = \inf_{n\in\mathbb{N}} \mu(A_n).$$

5. ***Countable sub-additivity.*** $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

Proof. Let $(X, \mathcal{F}, \mu)$ be a measure space, and $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{F}$.

1. Let $\{B_i : i \in \mathbb{N}\}$ such that $B_i = A_i$ for $i \in [n]$ and $B_i = \emptyset$ for $i \in \mathbb{N} \setminus [n]$. Then, $\{B_i : i \in \mathbb{N}\}$ is a countable collection of pair-wise disjoint sets, where $\cup_{i \in \mathbb{N}} B_i = \cup_{i=1}^n A_i$. Hence, it follows from the definition of measure, that

$$\mu(\bigcup_{i=1}^n A_i) = \mu(\bigcup_{i \in \mathbb{N}} B_i) = \sum_{i \in \mathbb{N}} \mu(B_i) = \sum_{i=1}^n \mu(A_i).$$

2. We can write $A_j = A_i \cup (A_j \setminus A_i)$, a finite union of disjoint sets. Therefore from finite additivity and non-negativity of measure, we have

$$\mu(A_j) = \mu(A_i) + \mu(A_j \setminus A_i) \geq \mu(A_i).$$

3. If $\{A_n : n \in \mathbb{N}\}$ is an increasing sequence of sets, then $\lim_{n \in \mathbb{N}} A_n = \cup_{n \in \mathbb{N}} A_n$. We construct a sequence of pair-wise disjoint sets $\{B_n : n \in \mathbb{N}\}$ inductively such that $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n > 1$. It follows that for all $n \in \mathbb{N}$,

$$\bigcup_{i \leq n} B_i = \bigcup_{i \leq n} A_i = A_n.$$

From σ-additivity of measures, it follows that

$$\begin{aligned} \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \mu(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \mu(B_n) = \lim_{n \in \mathbb{N}} \sum_{i=1}^n \mu(B_i) \\ &= \lim_{n \in \mathbb{N}} \mu(\bigcup_{i=1}^n B_i) = \lim_{n \in \mathbb{N}} \mu(A_n). \end{aligned}$$

4. If $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of sets, then we can form increasing sequence of sets $\{B_n : n \in \mathbb{N}\}$ such that $B_n = A_1 \setminus A_n$. From DeMorgan's law, we have

$$\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \left(\bigcup_{n \in \mathbb{N}} A_1 \setminus A_n\right) = \left(A_1 \setminus \left(\bigcap_{n \in \mathbb{N}} A_n\right)\right) = A_1 \setminus \left(\bigcap_{n \in \mathbb{N}} A_n\right).$$

Applying finite-additivity of measures to sets $A_1 \setminus \left(\bigcap_{n \in \mathbb{N}} A_n\right)$ and $\left(\bigcap_{n \in \mathbb{N}} A_n\right)$, we get

$$\mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \mu(A_1) - \mu\left(\bigcap_{n \in \mathbb{N}} A_n\right).$$

Now applying continuity from below to this increasing sequence of sets, we get

$$\mu(A_1) - \mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \mu\left(\bigcap_{n\in\mathbb{N}} B_n\right) = \lim_{n\in\mathbb{N}} \mu(B_n) = \lim_{n\in\mathbb{N}} \mu(A_1) - \mu(A_n).$$

Since $\mu(A_1)$ is finite, we can subtract it from both sides of the above equation to get the result.

5. We construct a pair-wise disjoint sequence of sets $\{B_n : n \in \mathbb{N}\}$ inductively from sets $\{A_n : n \in \mathbb{N}\}$, such that $B_n = A_n \setminus \cup_{i=1}^{n-1} A_i$. It follows from the construction that $\cup_{i=1}^{n} B_i = \cup_{i=1}^{n} A_i$, and from monotonicity that $\mu(B_n) \leq \mu(A_n)$ for all $n \in \mathbb{N}$. Hence, we conclude that

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n\in\mathbb{N}} \mu(B_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n).$$

□

Remark 1. We can replace finiteness of $\mu(A_1)$ by finiteness of $\mu(A_n)$ for some $n \in \mathbb{N}$. We can just take $B_j = A_n \setminus A_j$ for all $j \in \mathbb{N}$.

Remark 2. Finite assumption is necessary in continuity from above. It is possible that $\mu(A_j) = \infty$ for all $j \in \mathbb{N}$ and $\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) < \infty$.

2 Continuity of Measures

Definition 2.1. Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of subsets of a non-empty set X. We can define increasing sequence of sets $\{B_n : n \in \mathbb{N}\}$ and decreasing sequence of sets $\{C_n : n \in \mathbb{N}\}$, such that

$$B_n = \bigcup_{k\geq n} A_k \text{ and } C_n = \bigcap_{k\geq n} A_k.$$

Then, we can define **limit superior** and **limit inferior** of sets $\{A_n : n \in \mathbb{N}\}$ as

$$\limsup_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n, \qquad \liminf_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} C_n.$$

When $\limsup A_n = \liminf A_n$, we say that limit $\lim A_n$ of the sequences of set exists and

$$\lim A_n = \limsup A_n = \liminf A_n.$$

Remark 3. From the definition it is clear that the following hold.

$$\limsup_{n\in\mathbb{N}} A_n = \{x \in X : x \in A_n \text{ for infinitely many } n\}.$$
$$\liminf_{n\in\mathbb{N}} A_n = \{x \in X : x \in A_n \text{ for all but finitely many } n\}.$$

Proposition 2.2. *Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of increasing subsets of a non-empty set X. That is, $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$. Then,*

$$\liminf A_n = \limsup A_n = \bigcup_{n\in\mathbb{N}} A_n.$$

Proof. Let $B_n = \cup_{k\geq n} A_k = \cup_{k\in\mathbb{N}} A_k$ independent of n. Further, $C_n = \cap_{k\geq n} A_k = A_n$. Hence, we have

$$\liminf A_n = \bigcup_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n = \limsup A_n.$$

□

Proposition 2.3. *Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of decreasing subsets of a non-empty set X. That is, $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$. Then,*

$$\liminf A_n = \limsup A_n = \bigcap_{n\in\mathbb{N}} A_n.$$

Proof. Let $B_n = \cup_{k\geq n} A_k$, then $B_n = A_n$ since $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of sets. On the other hand, $C_n = \cap_{k\geq n} A_k = \cap_{k\in\mathbb{N}} A_k$, that is independent of n. Hence, we have

$$\liminf A_n = \bigcup_{n\in\mathbb{N}} C_n = \bigcap_{k\in\mathbb{N}} A_k = \limsup A_n.$$

□

Proposition 2.4. *Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of subsets of a non-empty set X. Then,*

$$\liminf A_n \subseteq \limsup A_n.$$

Proof. Let B_n and C_n be as in definition. Then, it is easy to see that $\{B_n : n \in \mathbb{N}\}$ and $\{C_n : n \in \mathbb{N}\}$ are decreasing and increasing sequences of sets respectively. Further, we have $C_n \subseteq A_m \subseteq B_m$ for all $m \geq n$. Further, since B_m is decreasing sequence of sets, we have for all $n \in \mathbb{N}$,

$$C_n \subseteq \bigcap_{m\geq n} B_m = \bigcap_{m\in\mathbb{N}} B_m = \limsup A_n.$$

Result follows from taking countable union of increasing sequence of sets $\{C_n : n \in \mathbb{N}\}$. □

Lemma 2.5 (Continuity of measures). *Let* $(X, \mathcal{F}, \mu)$ *be a finite measure space, and a sequence* $\{A_n \in \mathcal{F} : n \in \mathbb{N}\}$ *of measurable sets. Then,*

$$\mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n).$$

In particular, when $\lim A_n$ *exists then*

$$\mu(\lim A_n) = \lim \mu(A_n).$$

Proposition 2.6 (Borel-Cantelli Lemma I). *Let* $(X, \mathcal{F}, \mu)$ *be a measure space, and a sequence* $\{A_n \in \mathcal{F} : n \in \mathbb{N}\}$ *such that* $\sum_{n \in \mathbb{N}} \mu(A_n) < \infty$*. Then,*

$$\mu(\limsup_{n \in \mathbb{N}} A_n) = 0.$$

Proof. Let $B_n = \cup_{k \geq n} A_k$. Then, $\{B_n : n \in \mathbb{N}\}$ is a decreasing sequence of sets in $\mathcal{F}$ with $\limsup_n A_n = \cap_n B_n$. Hence, for each $n \in \mathbb{N}$, we have

$$\mu(\limsup_{n \in \mathbb{N}} A_n) \leq \mu(B_n).$$

From sub-additivity of measures, we have $\mu(B_n) \leq \sum_{k \geq n} \mu(A_k)$. Since, series $\sum_{n \in \mathbb{N}} \mu(A_n)$ converges, we have $\sum_{k \geq n} \mu(A_k)$ converging to zero as n grows large. $\square$