Lecture 24: Properties of Measures

1 Properties of Measures

We will assume $(X, \mathcal{F})$ to be the measurable space throughout this lecture, unless specified otherwise.

Definition 1.1. Let $(X, \mathcal{F})$ be a measurable space. A measure $\mu : \mathcal{F} \to [0, \infty]$ is called

1. a **probability measure**, if $\mu(X) = 1$,
2. a **finite measure**, if $\mu(X) < \infty$,
3. a **σ-finite measure**, if there exists a sequence $\{A_n \in \mathcal{F} : n \in \mathbb{N}\}$ such that $\cup_{n \in \mathbb{N}} A_n = X$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$,
4. a **semi-finite measure**, if for each $E \in \mathcal{F}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{F}$ with $F \subseteq E$ and $\mu(F) < \infty$,
5. **diffuse** or **atom-free**, if $\mu(\{x\}) = 0$, whenever $x \in X$ and $\{x\} \in \mathcal{F}$.

Lemma 1.2. *If μ is a finite measure, then $\mu(E)$ is finite for all $E \in \mathcal{F}$.*

Proof. Let $E \in \mathcal{F}$, then $X \setminus E \in \mathcal{F}$. Further, set X can be expressed as disjoint union of $E \sqcup (X \setminus E)$. Then, by finite additivity of measures, we have $\mu(X) = \mu(E) + \mu(X \setminus E)$. Hence, by non-negativity of measures, we have $\mu(E) < \infty$. □

Definition 1.3. Let $(X, \mathcal{F}, \mu)$ be a measure space. If $E = \cup_{j \in \mathbb{N}} E_j$ where $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$, then the set E is called of **σ-finite measure**.

Lemma 1.4. *A probability measure is finite. A finite measure is σ-finite.*

Proposition 1.5. *Every σ-finite measure is semi-finite.*

Proof. Let $(X, \mathcal{F}, \mu)$ be a measure space, with μ σ-finite. Since finite measures are trivial, we consider non-finite measures. Then, there exists a countable sequence of finite sets $\{A_n : n \in \mathbb{N}\}$ that cover X. We consider $E \in \mathcal{F}$ such that $\mu(E) = \infty$, and we have $E = \cup_{n\in\mathbb{N}} E \cap A_n$. Since, $\mu(E) = \infty$, set $M = \{n \in \mathbb{N} : E \cap A_n \neq \emptyset\}$ is non-empty. Let $m \in M$, then $E \cap A_m \subseteq E$ and is finite from monotonicity of measures and finiteness of A_n for each $n \in \mathbb{N}$. □

Remark 1. In practice, most measures are σ-finite. Non σ-finite measures have pathological properties.

Definition 1.6 (Uniform measure). Consider a finite set X with $\mathcal{F} = \mathcal{P}(X)$. We define a set function $\mu : \mathcal{F} \to [0, 1]$ by $\mu(A) = \frac{|A|}{|N|}$. Then, set function μ is a probability measure called the **uniform measure** on X.

Definition 1.7 (Dirac measure). For $x \in X$, we define the set function δ_x on $\mathcal{F}$ by

$$\delta_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Set function δ_x is called the **point mass at** x, or an **atom on** x, or the **Dirac function**.

Lemma 1.8. *Dirac function δ_x at x is probability measure on $(X, \mathcal{F})$. It is atom free only if $\{x\} \notin \mathcal{F}$.*

Definition 1.9 (Counting measure). We define a set function μ on $\mathcal{F}$ by

$$\mu(A) = \begin{cases} |A|, & A \text{ finite}, \\ \infty, & A \text{ infinite}. \end{cases}$$

Set function μ is called the **counting measure**.

Lemma 1.10. *Counting measure is finite iff X is finite set. It is never atom-free. It is probability measure iff $|X| = 1$.*

Example 1.11. Counting measure μ on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ is σ-finite, but not finite. Clearly, $\mu(X) = \infty$ and $\mu(A_n) = 1$ for pair-wise disjoint $A_n = \{n : n \in \mathbb{N}\}$, where $\cup_{n\in\mathbb{N}} A_n = \mathbb{N}$.

Example 1.12. Counting measure on $X = [0, 1]$ and $\mathcal{F} = \mathcal{B}_X$ is semi-finite but not σ-finite.

Definition 1.13. Let X be a non-empty set, $E \subseteq X$ be an infinite subset, and a function $f : X \to [0, \infty]$, then

$$\sum_{x \in E} f(x) = \sup_{F \subseteq E : F \text{ finite}} \sum_{x \in F} f(x).$$

Proposition 1.14. *Suppose that X is a finite/countable set. Then, each measure μ on $\mathcal{F} = \mathcal{P}(X)$ is of the form $\mu(A) = \sum_{x \in A} p(x)$ for some function $p : \mathcal{F} \to [0, \infty]$.*

Proof. We will show it for the case when $\mu(X) < \infty$. For each $x \in X$, since $\mathcal{P}(X)$ is theσ-algebra on X, all subsets of X are measurable. In particular, $\{x\}$ and $X \setminus \{x\}$ are measurable. Therefore, we have $\mu(\{x\}) = \mu(X) - \mu(X \setminus \{x\})$ from finite additivity of measures. We define $p(x) = \mu(\{x\})$ for all $x \in X$. Clearly, then $p(x) \geq 0$ from monotonicity of measures. Let $A \subseteq X$. Then, A is countable since X is countable. From the σ-additivity of countable pair-wise disjoint sets $\{x : x \in A\}$ that cover A, it follows that

$$\mu(A) = \mu(\bigcup_{x \in A} \{x\}) = \sum_{x \in A} \mu(\{x\}) = \sum_{x \in A} p(x).$$

□

Proposition 1.15. *Let $(X, \mathcal{P}(X))$ be a measurable space for non-empty set X. Then, any non-negative function $f : X \to [0, \infty]$ determines a measure $\mu : \mathcal{P}(X) \to [0, \infty]$ by $\mu(A) = \sum_{x \in A} f(x)$ for all $A \subseteq X$. This measure μ is*

1. *semi-finite iff $f(x) < \infty$ for all $x \in X$,*
2. *σ-finite iff μ is semi-finite and $\{x \in X : f(x) > 0\}$ is countable,*
3. *counting measure if $f(x) = 1$ for all $x \in X$,*
4. *dirac measure if for some $x_0 \in X$, we have $f(x) = \delta_{x_0}(\{x\})$ for all $x \in X$,*
5. *uniform measure if X finite and $f(x) = 1/|X|$ for all $x \in X$.*

Proof. It's easy to verify the null set has measure zero. It's also trivial to see σ-additivity holds for finite sets X. We will verify σ-additive property for general X. Consider $\{A_n \subseteq X : n \in \mathbb{N}\}$, a pair-wise disjoint sequence of subsets of X, and $E \subseteq X$ finite. Let $A = \cup_{n \in \mathbb{N}} A_n$, then $\{E \cap A_n \subseteq A_n : n \in \mathbb{N}\}$ is a pair-wise disjoint sequence of finite sets such that $E \cap A = \cup_{n \in \mathbb{N}} E \cap A_n$. Then, from σ-additivity of measure on finite sets, we have

$$\mu(E \cap A) = \sum_{n \in \mathbb{N}} \mu(E \cap A_n).$$

Therefore, we can conclude by taking supremums that

$$\begin{aligned}\mu(A) &= \sup_{E\subseteq A: E \text{ finite}} \sum_{x\in E} f(x) = \sup_{E\subseteq A: E \text{ finite}} \mu(E\cap A) = \sup_{E\subseteq A: E \text{ finite}} \sum_{n\in\mathbb{N}} \mu(E\cap A_n) \\ &= \sup_{E\subseteq A: E \text{ finite}} \sum_{n\in\mathbb{N}} \mu(E\cap A_n) = \sum_{n\in\mathbb{N}} \sup_{E\cap A_n\subseteq A_n: E \text{ finite}} \mu(E\cap A_n) = \sum_{n\in\mathbb{N}} \mu(A_n).\end{aligned}$$

Last step follows from the fact that for every finite E, only finitely many terms in the summation would be non-zero. Hence we can exchange supremum and summation.

1. If $f(x_0) = \infty$ for some $x_0 \in X$. Then, any set $\{x_0\}$ is not semi-finite. Further, if $f(x) < \infty$ for all $x \in X$. Then, for each set $A \subseteq X$, we can find a finite subset of A, hence $\mu(A) = \sum_{x\in A} f(x)$ is of finite measure.
2. It is clear that σ-finiteness implies semi-finiteness.
3. It follows trivially from the definition.
4. For any $A \subseteq X$, we see that $\mu(A) = \delta_{x_0}(A)$.
5. For any $A \subseteq X$, we see that $\mu(A) = \frac{|A|}{|X|}$.

□

Example 1.16. Consider an uncountable set X with σ-algebra $\mathcal{F}$ of countable or co-countable sets. That is,

$$\mathcal{F} = \{E \subseteq X : E \text{ countable or } X \setminus E \text{ countable }\}.$$

Then, the set function $\mu : \mathcal{F} \to [0, \infty]$ defined by

$$\mu(E) = \begin{cases} 0, & E \text{ countable}, \\ 1, & E \text{ co-countable}, \end{cases}$$

is a measure.

Example 1.17. Consider an infinite set X with σ-algebra $\mathcal{F} = \mathcal{P}(X)$. Then, the set function $\mu : \mathcal{F} \to [0, \infty]$ defined by

$$\mu(E) = \begin{cases} 0, & E \text{ finite}, \\ \infty, & E \text{ infinite}, \end{cases}$$

is finitely additive, but not a measure.

1.1 Complete Measure

Definition 1.18. Let $(X, \mathcal{F}, \mu)$ be a measure space. A set $N \in \mathcal{F}$ is said to be **μ-null** if $\mu(N) = 0$.

Lemma 1.19. *Any countable union of null sets is null.*

Proof. It follows from sub-additivity of measures. □

Definition 1.20. For a measure space $(X, \mathcal{F}, \mu)$, if a statement about points $x \in X$ is true except for x in some μ-null set, the statement is said to be true **μ-almost everywhere**.

Remark 2. If $\mu(E) = 0$ then $\mu(F) = 0$ for all $F \subseteq E$ by monotonicity if $F \in \mathcal{F}$. However, in general, $F \notin \mathcal{F}$.

Definition 1.21. Let $(X, \mathcal{F}, \mu)$ be measure space. Measure μ is called **complete** if its domain includes all subsets of null sets.

Theorem 1.22 (Measure completion). *Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{F} : \mu(N) = 0\}$ and*

$$\bar{\mathcal{F}} = \{E \cup F : E \in \mathcal{F}, F \subseteq N \text{ for some } N \in \mathcal{N}\}.$$

Then, $\bar{\mathcal{F}}$ is a σ-algebra, and there is a unique extension $\bar{\mu}$ of μ to a complete measure on $\bar{\mathcal{F}}$.

Proof. First, we will show that $\bar{\mathcal{F}}$ is a σ-algebra. Since $\mathcal{F}$ and $\mathcal{N}$ are closed under countable unions, so is $\bar{\mathcal{F}}$. To show $\bar{\mathcal{F}}$ is closed under complements, consider an element $E \cup F \in \bar{\mathcal{F}}$, where $E \in \mathcal{F}$ and $F \subseteq N \in \mathcal{N}$. We can assume that $E \cap N = \emptyset$, since otherwise we can replace F and N by $F \setminus E$ and $N \setminus E$ respectively. Then,

$$E \cup F = (E \cup N) \cap (X \setminus N \cup F).$$

Taking complements and using De Moivre's Theorem, we get

$$X \setminus (E \cup F) = X \setminus (E \cup N) \cup (N \setminus F),$$

where $E \cup N \in \mathcal{F}$ and $N \setminus F \subseteq N$ and hence $X \setminus (E \cup F)$ belongs to $\mathcal{F}$.

We can define a set function $\bar{\mu} : \bar{F} \to [0, \infty]$ for each $E \cup F \in \bar{F}$ as before, as $\bar{\mu}(E \cup F) = \mu(E)$. We verify that it is indeed a well-defined function by taking $E_1 \cup F_1 = E_2 \cup F_2$ where $E_j \in \mathcal{F}$ and $F_j \subseteq N_j \in \mathcal{N}$. Then, $E_1 \subseteq E_2 \cup N_2$, and hence $\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2)$ and similarly, $\mu(E_2) \leq \mu(E_1)$. It is easy to see that $\bar{\mu}$ is a complete measure on $\bar{\mathcal{F}}$, and unique measure that extends μ. □

Definition 1.23. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\bar{\mathcal{F}}$ and $\bar{\mu}$ as defined in measure completion theorem. Then, the unique extension $\bar{\mu}$ of measure μ is called the **completion** of μ, and $\bar{\mathcal{F}}$ is called the **completion of $\mathcal{F}$ with respect to μ.**