Lecture 25 : Integration of non-negative functions

1 Integration of non-negative functions

Definition 1.1. Let $(X, \mathcal{M}, \mu)$ be the measurable space, we define $L^+ = \{f \in \mathcal{F}(X, [0, \infty])\colon f \textit{ is } (\mu, B_{[0,\infty]}) \textit{ measurable}\}$.

Definition 1.2. If ϕ is simple, $\phi \in L^+$, with standard representation $\phi = \sum_{i=1}^n a_j \mathbb{1}_{E_j}$ then we define **integral** of ϕ with respect to μ by

$$\int \phi \; d\mu = \sum_{j=1}^{n} a_j \mu(E_j)$$

Remark 1. By convention, $0.\infty = 0$.

Remark 2. $\int \phi \; d\mu = \int \phi$.

Remark 3. $\int \phi \; d\mu = \int \phi(x) \; d\mu(x)$.

Remark 4. If $A \in \mathcal{M}$, then $\phi \mathbb{1}_A$ is simple.

$$\phi \mathbb{1}_A = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j \cap A}$$

$$\int_A \phi \; d\mu = \int \phi \mathbb{1}_A \; d\mu = \sum_{j=1}^{n} a_j \mu(E_j \cap A).$$

Remark 5. $\int_A \phi \; d\mu = \int \phi \mathbb{1}_A \; d\mu = \int_A \phi = \int \phi \mathbb{1}_A$.

Remark 6. $\int = \int_X$

Proposition 1.3. *Let ϕ, ψ simple in L^+,*

a) If $c \geq 0$, $\int c\phi \; d\mu = c \int \phi \; d\mu$.

b) $\int (\phi + \psi) \; d\mu = \int \phi \; d\mu + \int \psi \; d\mu$.

c) If $\phi \leq \psi$, then $\int \phi \; d\mu \leq \int \psi \; d\mu$.

d) The map $A \to \int_A \phi \, d\mu$ is measure on $\mathcal{M}$.

Proof. a) It is trivial. consider $\phi = \sum_{j=1}^m a_j \mathbb{1}_{E_j}$. Then $\int c\phi \, d\mu = \sum_{j=1}^n c a_j \mu(E_j) = c \sum_{j=1}^n a_j \mu(E_j) = c \int \phi \, d\mu$.

b) Let $\phi = \sum_{j=1}^m a_j \mathbb{1}_{E_j}$, $\psi = \sum_{k=1}^m b_j \mathbb{1}_{F_k}$. Then $\{E_j \cap f_k \colon j, k\}$ is pairwise disjoint and covers set X. Then

$$\phi + \psi = \sum_{j,k} (a_j + b_k) \mathbb{1}_{E_j \cap F_k}.$$

Where $\phi = \sum_{j,k} a_j \mathbb{1}_{E_j \cap F_k}$ and $\psi = \sum_{j,k} b_k \mathbb{1}_{E_j \cap F_k}$. To complete the proof consider the following

$$\begin{aligned} \int (\phi + \psi) \, d\mu = \sum_{j,k} (a_j + b_k) \; \mu(E_j \cap F_k) &= \sum_{j,k} a_j \; \mu(E_j \cap F_k) + \sum_{j,k} b_k \; \mu(E_j \cap F_k) \\ &= \int \phi \, d\mu + \int \psi \, d\mu. \end{aligned}$$

c) Given $\phi \le \psi$ then we can get $a_i \le b_k$ $\forall j, k$ such that $\mu(E_j \cap F_k) \ne 0$. Then

$$\begin{aligned} a_j \mu(E_j \cap F_k) &\le b_k \mu(E_j \cap F_k) \\ \sum_{j,k} a_j \mu(E_j \cap F_k) &\le \sum_{j,k} b_k \mu(E_j \cap F_k) \\ \int \phi \, d\mu &\le \int \psi \, d\mu \end{aligned}$$

d) Let $\{A_k \in \mathcal{M} \colon k \in \mathbb{N}\}$ be a disjoint sequence in $\mathcal{M}$, and call $A = \cup_{k \in \mathbb{N}} A_k$. Then

$$\begin{aligned} \int_A \phi \, d\mu &= \int \sum_{j=1}^n a_j \mathbb{1}_{E_j \cap A} \, d\mu \\ &= \sum_{j=1}^n a_j \; \mu(E_j \cap A) \\ &= \sum_{j=1}^n \sum_{k \in \mathbb{N}} a_j \; \mu(E_j \cap A_k) \\ &= \sum_{k \in \mathbb{N}} \int_{A_k} \phi \, d\mu \end{aligned}$$

Which completes the proof.

□

Definition 1.4. For $f \in L^+$ we can define $\int f \, d\mu = \sup\{\int \phi \, d\mu \colon 0 \leq \phi \leq f, \phi$ being simple function$\}$.

Remark 7. This definition is consistent for simple functions.

Remark 8. If $f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$.

Theorem 1.5 (Monotone convergence theorem). *Let* $\{f_n \in L^+ \colon n \in \mathbb{N}\}$ *such that* $f_j(x) = f_{j+1}(x) \ \forall j \in \mathbb{N}$, *and* $f(x) = \lim_{j \in \mathbb{N}} f_j(x) = \sup_{j \in \mathbb{N}} f_j(x)$, *then* $\lim_{n \in \mathbb{N}} \int f_n \, d\mu = \int (\lim_{n \in \mathbb{N}} f_n) \, d\mu$

Proof. Let $x \in X$ and $\{f_n(x) \colon n \in \mathbb{N}\}$, increasing sequence with $f = \lim_{n \to \infty} f_n$ exists and possibly ∞ and $f_n \leq f$. Then from the previous remark we know that $\int f_n \, d\mu \leq \int f \, d\mu \ \forall n \in \mathbb{N}$. To prove reverse inequality consider $\lim_{n \to \infty} \int f_n \, d\mu \leq \int f \, d\mu \ \forall n \in \mathbb{N}$. Let $\alpha \in (0,1)$ and $0 \leq \phi \leq f$ simple function, $E_n = \{x \in X \colon f_n(x) \geq \alpha \phi(x)\}$. Then $\{E_n \colon n \in \mathbb{N}\}$ is a sequence of increasing sets such that $\cup_{n \in \mathbb{N}} = X$. Then $\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \alpha \int_{E_n} \phi \, d\mu$. Then from previous proposition part d and the continuity from below definition we get $\lim_{n \to \infty} \int_{E_n} \phi \, d\mu = \int_X \phi \, d\mu$. Hence $\alpha \int \phi \, d\mu \leq \lim_{n \in \mathbb{N}} \int f_n \, d\mu$. Then $\int \phi \, d\mu \leq \lim_{n \in \mathbb{N}} \int f_n \, d\mu$. Then by taking supremum over all such simple functions we get $\int f \, d\mu \leq \lim_{n \in \mathbb{N}} \int f_n \, d\mu$ which completes the proof. □

Theorem 1.6. *If* $\{f_n \in L^+\}$ *and* $f = \sum_n f_n$ *then* $\int f \, d\mu = \sum_n \int f_n \, d\mu$.

Proof. Consider $f_1, f_2 \in L^+$, and $\{\phi_j\}$ and ψ_j increasing and converging to f_1, f_2 respectively. Then

$$\begin{aligned}
\int (f_1 + f_2) \, d\mu &= \lim_{j \in \mathbb{N}} \int (\phi_j + \psi_j) \\
&= \lim_{j \in \mathbb{N}} \int \phi_j + \lim_{j \in \mathbb{N}} \int \psi_j \\
&= \int f_1 \, d\mu + \int f_2 \, d\mu
\end{aligned}$$

Hence by induction we get $\int \sum_{i=1}^{n} f_i \, d\mu = \sum_{i=1}^{n} \int f_i \, d\mu$ for any any finite n. From Monotone convergence theorem, as $n \to \infty$ we get $\int \sum_{i=1}^{\infty} f_i \, d\mu = \sum_{i=1}^{\infty} \int f_i \, d\mu$. which is $\int f \, d\mu = \sum_n \int f_n \, d\mu$. □

Proposition 1.7. *If* $f \in L^+$, *and* $\int f \, d\mu = 0$,*then* $f = 0$ *a.e*

Proof. This is clear if f is simple. Since, if $f = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j}$ then $\int f \, d\mu = 0$ if and only if either $a_j = 0$ or $\mu(E_j) = 0$. In general, ϕ simple, $0 \leq \phi \leq f$ and $f = 0$ *a.e* then $\phi = 0$ *a.e*. Hence, $\int f \, d\mu = \sup_\phi \int \phi \, d\mu = 0$. Conversely, $\{x \colon f(x) > 0\} = \cup_{n \in \mathbb{N}} E_n$ where $E_n = \{x \in X \colon f(x) > \frac{1}{n}\}$. If $f \neq 0$ *a.e* then $\exists n \in \mathbb{N}$ such that $\mu(E_n) > 0$. That is, $\mu\{x \in X \colon f(x) > \frac{1}{n}\} > 0$. Then $\int f \, d\mu \geq \int_{E_n} f \, d\mu > \frac{1}{n} \mu(E_n) > 0$. Which is contradiction since we considered $\int f \, d\mu = 0$. □

Corollary 1.8. *If* $\{f_n \in L^+ : n \in \mathbb{N}\}$, $f \in L^+$, *such that* $\sup_{n\to\infty} f_n(x) = f(x)$ *for* $a.e\ X$ *then* $\lim_n \int f_n \, d\mu = \int f \, d\mu$.

Proof. Let $E = \{x \in X : f_n(x) \uparrow f(x)\}$ where $\mu(E^c) = 0$. Then $f - f\mathbb{1}_E = 0$ $a.e$ and $f_n - f_n\mathbb{1}_E = 0$ $a.e.$ By Monotone convergence theorem, $\int f \, d\mu = \int f\mathbb{1}_E \, d\mu = \lim \int f_n\mathbb{1}_E \, d\mu = \lim \int f_n \, d\mu$. □

Lemma 1.9 (Fatou's lemma). *If* $\{f_n \in L^+ : n \in \mathbb{N}\}$ *then* $\int (\liminf f_n) \, d\mu \leq \liminf_n \int f_n \, d\mu$.

Proof. For $k \in \mathbb{N}$, $\inf_{n\geq k} f_k \leq f_j \ \forall j \geq k$. Hence, $\int \inf_{n\geq k} f_n d\mu \leq \inf_{j\geq k} \int f_j \, d\mu$. By letting $n \to \infty$ and applying monotone convergence theorem we get $\int \liminf_n f_n \leq \liminf_n \int f_n \, d\mu$. □