1 Integration of non-negative functions

Definition 1.1. Let \((X, \mathcal{M}, \mu)\) be the measurable space, we define \(L^+ = \{ f \in \mathcal{F}(X, [0, \infty]) : \text{\textit{f is \textit{\mu}}, \textit{\mathcal{B}_{[0,\infty)}} \textit{measurable}\}\}.

Definition 1.2. If \(\phi\) is simple, \(\phi \in L^+\), with standard representation \(\phi = \sum_{i=1}^{n} a_j \mathbb{1}_{E_j}\), then we define \textit{integral} of \(\phi\) with respect to \(\mu\) by

\[
\int \phi \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j)
\]

Remark 1. By convention, \(0 \cdot \infty = 0\).

Remark 2. \(\int \phi \, d\mu = \int \phi\).

Remark 3. \(\int \phi \, d\mu = \int \phi(x) \, d\mu(x)\).

Remark 4. If \(A \in \mathcal{M}\), then \(\phi \mathbb{1}_A\) is simple.

\[
\phi \mathbb{1}_A = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j \cap A}
\]

\[
\int_A \phi \, d\mu = \int \phi \mathbb{1}_A \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j \cap A).
\]

Remark 5. \(\int_A \phi \, d\mu = \int \phi \mathbb{1}_A \, d\mu = \int_A \phi = \int \phi \mathbb{1}_A\).

Remark 6. \(\int = \int_X\).

Proposition 1.3. Let \(\phi, \psi\) \textit{simple in } \(L^+\),

a) If \(c \geq 0\), \(\int c \phi \, d\mu = c \int \phi \, d\mu\).

b) \(\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu\).

c) If \(\phi \leq \psi\), then \(\int \phi \, d\mu \leq \int \psi \, d\mu\).
d) The map $\mu : A \to \int \phi \ d\mu$ is measure on $\mathcal{M}$.

Proof. a) It is trivial. Consider $\phi = \sum_{j=1}^{m} a_j 1_{E_j}$. Then $\int c \phi \ d\mu = \sum_{j=1}^{n} c a_j \mu(E_j) = c \sum_{j=1}^{n} a_j \mu(E_j) = c \int \phi \ d\mu$.

b) Let $\phi = \sum_{j=1}^{m} a_j 1_{E_j}$, $\psi = \sum_{k=1}^{n} b_j 1_{F_k}$. Then $\{E_j \cap F_k : j,k\}$ is pairwise disjoint and covers set $X$. Then

$$\phi + \psi = \sum_{j,k} (a_j + b_k) 1_{E_j \cap F_k}.$$

Where $\phi = \sum_{j,k} a_j 1_{E_j \cap F_k}$ and $\psi = \sum_{j,k} b_k 1_{E_j \cap F_k}$. To complete the proof consider the following

$$\int (\phi + \psi) \ d\mu = \sum_{j,k} (a_j + b_k) \mu(E_j \cap F_k) = \sum_{j,k} a_j \mu(E_j \cap F_k) + \sum_{j,k} b_k \mu(E_j \cap F_k)$$

$$= \int \phi \ d\mu + \int \psi \ d\mu.$$

c) Given $\phi \leq \psi$ then we can get $a_i \leq b_k \forall j,k$ such that $\mu(E_j \cap F_k) \neq 0$. Then

$$a_j \mu(E_j \cap F_k) \leq b_k \mu(E_j \cap F_k)$$

$$\sum_{j,k} a_j \mu(E_j \cap F_k) \leq \sum_{j,k} b_k \mu(E_j \cap F_k)$$

$$\int \phi \ d\mu \leq \int \psi \ d\mu$$

d) Let $\{A_k \in \mathcal{M} : k \in \mathbb{N}\}$ be a disjoint sequence in $\mathcal{M}$, and call $A = \bigcup_{k \in \mathbb{N}} A_k$. Then

$$\int \phi \ d\mu = \int \sum_{j=1}^{n} a_j 1_{E_j \cap A} \ d\mu$$

$$= \sum_{j=1}^{n} a_j \mu(E_j \cap A)$$

$$= \sum_{j=1}^{n} \sum_{k \in \mathbb{N}} a_j \mu(E_j \cap A_k)$$

$$= \sum_{k \in \mathbb{N}} \int_{A_k} \phi \ d\mu$$

Which completes the proof. \[\square\]
Definition 1.4. For $f \in L^+$ we can define $\int f \, d\mu = \sup \{ \int \phi \, d\mu : 0 \leq \phi \leq f, \phi$ being simple function}. 

Remark 7. This definition is consistent for simple functions.

Remark 8. If $f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$.

Theorem 1.5 (Monotone convergence theorem). Let $\{f_n \in L^+ : n \in \mathbb{N}\}$ such that $f_j(x) = f_{j+1}(x) \ \forall j \in \mathbb{N}$, and $f(x) = \lim_{j \in \mathbb{N}} f_j(x) = \sup_{j \in \mathbb{N}} f_j(x)$, then 

$\lim_{n \in \mathbb{N}} \int f_n \, d\mu = \int (\lim_{n \in \mathbb{N}} f_n) \, d\mu$

Proof. Let $x \in X$ and $\{f_n(x) : n \in \mathbb{N}\}$, increasing sequence with $f = \lim_{n \to \infty} f_n$ exists and possibly $\infty$ and $f_n \leq f$. Then from the previous remark we know that $\int f_n \, d\mu \leq \int f \, d\mu \ \forall n \in \mathbb{N}$. To prove reverse inequality consider $\lim_{n \to \infty} \int f_n \, d\mu \leq \int f \, d\mu \ \forall n \in \mathbb{N}$. Let $\alpha \in (0,1)$ and $0 \leq \phi \leq f$ simple function, $E_n = \{x \in X : f_n(x) \geq \alpha \phi(x)\}$. Then $\{E_n : n \in \mathbb{N}\}$ is a sequence of increasing sets such that $\cup_{n \in \mathbb{N}} E = X$. Then $\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \alpha \int_{E_n} \phi \, d\mu$. Then from previous proposition part d and the continuity from below definition we get $\lim_{n \to \infty} \int_{E_n} \phi \, d\mu = \int_X \phi \, d\mu$. Hence $\alpha \int \phi \, d\mu \leq \lim_{n \in \mathbb{N}} \int f_n \, d\mu$. Then $\int \phi \, d\mu \leq \lim_{n \in \mathbb{N}} \int f_n \, d\mu$. Then by taking supremum over all such simple functions we get $\int f \, d\mu \leq \lim_{n \in \mathbb{N}} \int f_n \, d\mu$ which completes the proof. 

Theorem 1.6. If $\{f_n \in L^+\}$ and $f = \sum f_n$ then $\int f \, d\mu = \sum_n \int f_n \, d\mu$.

Proof. Consider $f_1, f_2 \in L^+$, and $\{\phi_j\}$ and $\psi_j$ increasing and converging to $f_1, f_2$ respectively. Then 

$$\int (f_1 + f_2) \, d\mu = \lim_{j \in \mathbb{N}} \int (\phi_j + \psi_j)$$

$$= \lim_{j \in \mathbb{N}} \int \phi_j + \lim_{j \in \mathbb{N}} \int \psi_j$$

$$= \int f_1 \, d\mu + \int f_2 \, d\mu$$

Hence by induction we get $\int \sum_{i=1}^n f_i \, d\mu = \sum_{i=1}^n \int f_i \, d\mu$ for any any finite $n$. From Monotone convergence theorem, as $n \to \infty$ we get $\int \sum_{i=1}^\infty f_i \, d\mu = \sum_{i=1}^\infty \int f_i \, d\mu$. which is $\int f \, d\mu = \sum_n \int f_n \, d\mu$. 

Proposition 1.7. If $f \in L^+$, and $\int f \, d\mu = 0$, then $f = 0$ a.e

Proof. This is clear if $f$ is simple. Since, if $f = \sum_{j=1}^n a_j 1_{E_j}$ then $\int f \, d\mu = 0$ if and only if either $a_j = 0$ or $\mu(E_j) = 0$. In general, $\phi$ simple, $0 \leq \phi \leq f$ and $f = 0$ a.e then $\phi = 0$ a.e. Hence, $\int f \, d\mu = \sup \phi \int \phi \, d\mu = 0$. Conversely, $\{x : f(x) > 0\} = \cup_{n \in \mathbb{N}} E_n$ where $E_n = \{x \in X : f(x) > \frac{1}{n}\}$. If $f \not= 0$ a.e then $\exists n \in \mathbb{N}$ such that $\mu(E_n) > 0$. That is, $\mu(x \in X : f(x) > \frac{1}{n}) > 0$. Then $\int f \, d\mu \geq \int_{E_n} f \, d\mu > \frac{1}{n} \mu(E_n) > 0$. Which is contradiction since we considered $\int f \, d\mu = 0$. 


Corollary 1.8. If \( \{ f_n \in L^+: n \in \mathbb{N} \} \), \( f \in L^+ \), such that \( \sup_{n \to \infty} f_n(x) = f(x) \) for a.e \( X \) then \( \lim_n \int f_n \, d\mu = \int f \, d\mu \).

Proof. Let \( E = \{ x \in X: f_n(x) \uparrow f(x) \} \) where \( \mu(E^c) = 0 \). Then \( f - f1_E = 0 \) a.e and \( f_n - f_n1_E = 0 \) a.e. By Monotone convergence theorem, \( \int f \, d\mu = \int f1_E \, d\mu = \lim \int f_n1_E \, d\mu = \lim \int f_n \, d\mu \).

Lemma 1.9 (Fatou’s lemma). If \( \{ f_n \in L^+: n \in \mathbb{N} \} \) then \( \int (\liminf f_n) \, d\mu \leq \liminf \int f_n \, d\mu \).

Proof. For \( k \in \mathbb{N} \), \( \inf_{n \geq k} f_n \leq f_j \forall j \geq k \). Hence, \( \int \inf_{n \geq k} f_n \, d\mu \leq \inf_{j \geq k} \int f_j \, d\mu \).
By letting \( n \to \infty \) and applying monotone convergence theorem we get \( \int \liminf \int f_n \, d\mu \leq \liminf \int f_n \, d\mu \). \( \square \)