# Lecture 26: Dominated Convergence Theorem

*Continuation of Fatou's Lemma.*

**Corollary 0.1.** *If $f \in L^+$ and $\{f_n \in L^+ : n \in \mathbb{N}\}$ is any sequence of functions such that $f_n \to f$ almost everywhere, then*

$$\int_X f \leqslant \liminf \int_X f_n.$$

*Proof.* Let $f_n \to f$ everywhere in $X$. That is, $\liminf f_n(x) = f(x)$ $(= \limsup f_n$ also) for all $x \in X$. Then, by Fatou's lemma,

$$\int_X f = \int_X \liminf f_n \leqslant \liminf \int_X f_n.$$

If $f_n \nrightarrow f$ everywhere in $X$, then let $E = \{x \in X : \liminf f_n(x) \neq f(x)\}$. Since $f_n \to f$ almost everywhere in $X$, $\mu(E) = 0$ and

$$\int_X f = \int_{X-E} f \quad \text{and} \quad \int_X f_n = \int_{X-E} f_n \ \ \forall n$$

thus making $f_n \nrightarrow f$ everywhere in $X - E$. Hence,

$$\int_X f \quad = \quad \int_{X-E} f \quad \leqslant \quad \liminf \int_{X-E} f_n \quad = \quad \liminf \int_X f_n.$$

$\square$

**Example 0.2 (Strict inequality).** Let $S_n = [n, n+1] \subset \mathbb{R}$ and $f_n = \chi_{S_n}$. Then, $f = \liminf f_n \ = \ 0$ and

$$0 \ = \ \int_{\mathbb{R}} f \, d\mu \ < \ \liminf \int_{\mathbb{R}} f_n \, d\mu \ = \ 1.$$

**Example 0.3 (Importance of non-negativity).** Let $S_n = [n, n+1] \subset \mathbb{R}$ and $f_n = -\chi_{S_n}$. Then, $f = \liminf f_n \ = \ 0$ but

$$0 \ = \ \int_{\mathbb{R}} f \, d\mu \ > \ \liminf \int_{\mathbb{R}} f_n \, d\mu \ = \ -1.$$

**Proposition 0.4.** *If $f \in L^+$ and $\int_X f\, d\mu < \infty$, then (a) the set $A = \{x \in X : f(x) = \infty\}$ is a null set and (b) the set $B = \{x \in X : f(x) > 0\}$ is $\sigma$-finite.*

*Proof.* Recall that for any $f \in L^+$

$$\int_X f\, d\mu \;=\; \sup\left\{\int_X \phi\, d\mu \;:\; 0 \leqslant \phi \leqslant f,\ \phi \text{ simple}\right\}$$

and for a simple function $\psi$, $\int_X \psi\, d\mu \;=\; \sum_{j=1}^{n} a_j \mu(E_j)$, where $E_j = \psi^{-1}(\{a_j\})$ and $\{a_1, a_2, \ldots, a_n\}$ is the the range of $\psi$.

*(a)* Assume, on the contrary, that $\mu(A) > 0$ and let $I = \int_X f\, d\mu \;<\; \infty$. Define a simple function $\phi$ as

$$\phi = 2\frac{I}{\mu(A)}\chi_A.$$

Since $f(x) = \infty$ for all $x \in A$ and $\phi(x) = 0$ for all $x \in X - A$, $\phi(x) \leqslant f(x)$ for all $x \in X$ and therefore, $\int_X \phi\, d\mu \leqslant \int_X f\, d\mu$. But $\int_X \phi\, d\mu = 2I > I = \int_X f\, d\mu$, which is a contradiction. Thus, $\mu(A) = 0$.

*(b)* A $\sigma$-finite set is a countable union of sets of finite measure. Define $B_n = \{x \in X \;:\; f(x) > n^{-1}\}$. Then, $B$ is a countable union of $B_n$s. For each $B_n$, define the simple functions $\phi_n = n^{-1}\chi_{B_n}$. For all $n \in \mathbb{N}$, $\phi_n \leqslant f$, and hence $\int_X \phi_n\, d\mu \;\leqslant\; \int_X f\, d\mu \;<\; \infty$. Since $\int_X \phi_n\, d\mu \;=\; n^{-1}\mu(B_n)$, $\mu(B_n) < \infty$. Thus, $B = \cup_{n\in\mathbb{N}} B_n$ is $\sigma$-finite.

$\square$

# 1 Integration of Real-Valued Functions

We now discuss integration of real-valued function which need not be positive. Let $f^+$ and $f^-$ be the positive and negative parts of $f$ respectively, where

$$f^+(x) = \max\{f(x), 0\} \quad \text{and} \quad f^-(x) = \max\{-f(x), 0\} \quad \text{for all} \quad x \in X.$$

Then, $f \;=\; f^+ - f^-$. Note that both $f^+$ and $f^-$ are positive real-valued functions. If at least one of $\int_X f^+\, d\mu$ and $\int_X f^-\, d\mu$ is finite, then we define

$$\int_X f\, d\mu \;=\; \int_X f^+\, d\mu \;-\; \int_X f^-\, d\mu.$$

If both $\int_X f^+\, d\mu$ and $\int_X f^-\, d\mu$ are finite, then $f$ is said to be **integrable**. Since $|f| = f^+ + f^-$, $f$ is integrable iff $\int_X |f|\, d\mu < \infty$.

**Proposition 1.1.** *The set of integrable real-valued functions on $X$, denoted by $\mathcal{F}(X,\mathbb{R})$, is a real vector space, and the integral is linear functional on it.*

*Proof.* To prove that $\mathcal{F}(X,\mathbb{R})$ is a vector space, it suffices to prove that any linear combination of integrable real-valued functions in $\mathcal{F}(X,\mathbb{R})$ also in $\mathcal{F}(X,\mathbb{R})$. For any $f,g \in \mathcal{F}(X,\mathbb{R})$ and $a,b \in \mathbb{R}$, $|af+bg| \leqslant |a||f|+|b||g|$ (by triangle inequality). Hence, $\int_X |af+bg|\,d\mu \leqslant \int_X (|a||f|+|b||g|)\,d\mu = |a|\int_X |f|\,d\mu + |b|\int_X |g|\,d\mu < \infty$ since both $f$ and $g$ are integrable. Thus, $af+bg$ is also in $\mathcal{F}(X,\mathbb{R})$.

To prove that the functional $I : f \longmapsto \int_X f\,d\mu$, $f \in \mathcal{F}(X,\mathbb{R})$, is linear, we need to show that *(a)*$I(cf) = cI(f)$ and *(b)*$I(f+g) = I(f)+I(g)$ for all $f,g \in \mathcal{F}(X,\mathbb{R})$.

*(a)* We will use the facts that $(cf)^+ = cf^+$ and $(cf)^- = cf^-$ for $c \geqslant 0$ and that $(cf)^+ = |c|f^-$ and $(cf)^- = |c|f^+$ for $c < 0$. Recall that for any $g \in L^+$, $I(cg) = cI(g)$ and for any $f \in \mathcal{F}(X,\mathbb{R})$, both $f^+$ and $f^-$ are positive. Let $c \geqslant 0$. Then, using the above facts, $I(cf) = c(I(f^+) - I(f^-)) = cI(f)$. For $c < 0$, $I(cf) = I((cf)^+) - I((cf)^-) = |c|(I(f^-) - I(f^+)) = -|c|I(f) = cI(f)$.

*(b)* Let $f,g \in \mathcal{F}(X,\mathbb{R})$ and $h = f+g$. Then, $h^+ - h^- = f^+ - f^- + g^+ - g^-$ and consequently $h^+ + f^- + g^- = h^- + f^+ + g^+$. Recall that if $\{f_n\}$ is a finite of infinite sequence in $L^+$ and $f = \sum_n f_n$, then $\int f = \sum_n \int f_n$. So,

$$\int h^+ + \int f^- + \int g^- \;=\; \int h^- + \int f^+ + \int g^+.$$

Rearranging the terms above, we get

$$\int h \;=\; \int h^+ - \int h^- \;=\; \int f^+ - \int f^- + \int g^+ - \int g^- \;=\; \int f + \int g.$$

□

**Proposition 1.2.** *For any $f \in \mathcal{F}(X,\mathbb{R})$, $|\int f| \leqslant \int |f|$.*

*Proof.* If $\int f = 0$, then this is trivial. For any real $f$, $|\int f| = |\int f^+ - \int f^-| \leqslant |\int f^+| + |\int f^-| = \int f^+ + \int f^- = \int |f|$ (by triangle inequality). □

**Proposition 1.3.** *(a) For any $f \in \mathcal{F}(X,\mathbb{R})$, $A = \{x : f(x) \neq 0\}$ is $\sigma$-finite.*

*(b) If $f,g \in \mathcal{F}(X,\mathbb{R})$, then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ iff $\int |f-g| = 0$ iff $f = g$ almost everywhere.*

*Proof. (a)* Note that $A = A^+ \cup A^-$, where $A^+ = \{x : f^+(x) > 0\}$ and $A^- = \{x : f^-(x) > 0\}$. Since both $f^+$ and $f^-$ are in $L^+$, by Proposition 0.4, both $A^+$ and $A^-$ are $\sigma$-finite. Hence, $A$ is $\sigma$-finite.

*(b)* The second equivalence follows from the fact that for any $h \in L^+$, $\int h = 0$ iff $h = 0$ almost everywhere. If $\int |f - g| = 0$, then by Proposition 1.2, for any $E \in \mathcal{M}$,

$$\left| \int_E f - \int_E g \right| \leqslant \int_X \chi_E |f - g| \leqslant \int_X |f - g| = 0$$

so that $\int_E f = \int_E g$. Let $h = f - g$ and assume that $f = g$ almost everywhere is false, then at least one of $h^+$ and $h^-$ must be nonzero on a set of positive measure. Let $E = \{x : h^+(x) > 0\}$ be one such set; note that $h^-(x) = 0$ and hence, $\int_E h^-(x) = 0$ for all $x \in E$. Then, $\int_E f - \int_E g = \int_E h = \int_E h^+ > 0$. Similar conclusion can be drawn for $h^-$ being nonzero on a set of positive measure.

□

*Remark* 1. *(i)* Altering functions on a mull set does not alter their integration.

*(ii)* Let $E \in \mathcal{M}$. Then, it is possible to integrate $f$ by defining $f|_{E^c} = 0$.

*(iii)* It is possible to treat $\overline{\mathbb{R}}$-valued functions that are finite almost everywhere as $\mathbb{R}$-valued functions.

**Definition 1.4.** $L^1$ can be redefined as follows:

$L^1(\mu) = \{$Equivalence class of almost everywhere-defined integrable functions on $X\}$,

where two functions $f$ and $g$ are equivalent if $\mu(\{x \in X : f(x) \neq g(x)\}) = 0$.

*Remark* 2. *(i)* $L^1(\mu)$ is still a vector space (under pointwise almost everywhere addition and scalar multiplication).

*(ii)* $f \in L^1(\mu)$ will mean that $f$ is an almost everywhere-defined integrable function.

*(iii)* For any two $f, g \in L^1(\mu)$, define $\rho(f, g) = \int |f - g|\, d\mu$. This is a metric, since it is symmetric, satisfies triangle inequality, and is 0 if $f$ and $g$ are equal almost everywhere. This definition allows $L^1(\mu)$ to be a metric space with $\rho(f, g)$ as the metric.

**Theorem 1.5 (The Dominated Convergence Theorem).** *Let $\{f_n \in L^1 : n \in \mathbb{N}\}$ be a sequence of functions such that (a) $f_n \to f$ almost everywhere and (b) there exists a non-negative $g \in L^1$ such that $|f_n| \leqslant g$ almost everywhere for all $n \in \mathbb{N}$. Then, $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.*

*Remark* 3. *(i)* $\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$ is an equivalent statement.

*(ii)* Here $g$ **dominates** $f_n$s.

*Proof.* Since $f$ is the limit of measurable functions $\{f_n\}$ almost every where, it is measurable. Since $|f_n| \leqslant g$ almost everywhere, $|f| = \lim_{n\to\infty} f_n \leqslant g$ almost everywhere, and hence, $f \in L^1$. Furthermore, $g + f_n \leqslant 0$ almost everywhere and $g - f_n \leqslant 0$ almost everywhere. By Corollary 0.1

$$\int g + \int f = \int (g+f) \leqslant \liminf \int (g+f_n) = \int g + \liminf \int f_n,$$

or

$$\int f \leqslant \liminf \int f_n. \tag{1}$$

Using Corollary 0.1 for $g - f_n$ we obtain

$$\int g - \int f = \int (g-f) \leqslant \liminf \int (g-f_n) = \int g - \liminf \int f_n,$$

or

$$\int f \geqslant \limsup \int f_n. \tag{2}$$

Since $\liminf \int f_n \leqslant \limsup \int f_n$, using 1 and 2 we get

$$\int f \leqslant \liminf \int f_n \leqslant \limsup \int f_n \leqslant \int f,$$

which forces

$$\int f = \liminf \int f_n = \limsup \int f_n \leqslant \int f = \lim_{n\to\infty} \int f_n.$$

as claimed. □

**Theorem 1.6.** *Let $\{f_n \in L^1 : n \in \mathbb{N}\}$ be a sequence of functions such that $\sum_{n\in\mathbb{N}} \int |f_n| < \infty$. Then, $\sum_{n\in\mathbb{N}} f_n$ converges to a function in $L^1$ and $\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n$.*

*Proof.* Recall that if $\{h_n\}$ is a finite of infinite sequence in $L^+$, then $\int \sum_n h_n = \sum_n \int h_n$. Set $h_n = |f_n|$ and let $g = \sum_{n\in\mathbb{N}} |f_n|$. Then, $\int g = \sum_{n\in\mathbb{N}} \int |f_n| < \infty$ and hence $g \in L^1$.

By Proposition 0.4, $g(x)(= \sum_{n\in\mathbb{N}} |f_n(x)|)$ is finite for all $\{x : g(x) > 0\}$, and for each such $x$ $\sum_{n\in\mathbb{N}} f_n(x)$ converges. Furthermore, the partial sums $F_k \triangleq$

$\sum_{n=1}^{k} f_n \leqslant g$ (by triangle inequality) for all $k$. We can now apply dominated convergence theorem to the sequence of partial sums $F_k$ to obtain

$$\int \lim_{k\to\infty} F_k = \lim_{k\to\infty} \int F_k,$$

which can be simplified to (using linearity of $\int$)

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n.$$

□

**Theorem 1.7.** *If $f \in L^1$ and $\epsilon > 0$, then there is an integrable simple function $\phi = \sum a_j \chi_{E_j}$ such that $\int |f - \phi| < \epsilon$. (That is, the integrable simple functions are dense in $L^1$ in the $L^1$ metric.)*

*Proof.* Recall that for any real-valued measurable function $g$, there exists a sequence $\{\psi_n\}$ of simple functions such that $\psi_n \to g$ and $0 \leqslant |\psi_1| \leqslant |\psi_2| \leqslant \ldots \leqslant |g|$ pointwise. Let $\{\phi_n\}$ be as above for $f$. Then, $\phi_n$s are integrable. Since $|\phi_n - f| \leqslant 2|f|$, $\int |\phi_n - f| < \epsilon$ for sufficiently large $n$ by the dominated convergence theorem. □

# 2 Modes of Convergence

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of functions in $L^1$ and $f \in L^1$.

**Definition 2.1 (Convergence in $L^1$).** If $f_n \to f$ in the metric $\rho(f,g) = \int |f - g|\, d\mu$, then $\{f_n\}$ is said to converge to $f$ in $L^1(\mu)$.

**Lemma 2.2.** *$f_n \to f$ in $L^1$ iff $\lim_{n\in\mathbb{N}} \int |f_n - f|\, d\mu = 0$.*

**Definition 2.3 (Pointwise Convergence).** $\{f_n\}$ is said to converge to $f$ pointwise if $f_n(x)$ converges to $f(x)$ for all $x \in X$. In other words, for every $\epsilon > 0$ and $x$, there exists an $N_{\epsilon,x}$ such that $|f_n(x) - f(x)| \leqslant \epsilon$ for all $n \geqslant N_{\epsilon,x}$.

**Definition 2.4 (Uniform Convergence).** $\{f_n\}$ is said to converge to $f$ uniformly if for every $\epsilon > 0$, there exists an $N_\epsilon$ such that $|f_n(x) - f(x)| \leqslant \epsilon$ for all $n \geqslant N_\epsilon$ and $x \in X$.

**Definition 2.5 (Almost Everywhere Convergence).** $\{f_n\}$ is said to converge to $f$ almost everywhere if $\mu(\{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0$.

**Definition 2.6 (Convergence in Measure).** $\{f_n\}$ is said to converge to $f$ in measure if for every $\epsilon > 0$, $\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geqslant \epsilon\}) = 0$.

**Definition 2.7 (Cauchy Convergence).** $\{f_n\}$ is said to be Cauchy in measure if for every $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f_m(x)| \geqslant 0\}) \to 0$ as $m, n \to \infty$.

**Theorem 2.8.** *If $f_n \to f$ almost everywhere and $f_n \leqslant g$ for all $n \in \mathbb{N}$ and some $g \in L^1$, then $f_n \to f$ in $L^1$.*

*Proof.* Follows from the dominated convergence theorem since $|f_n - f| \leqslant 2g$. $\square$

**Proposition 2.9.** *If $f_n \to f$ in $L^1$, then $f_n \to f$ in measure.*

*Proof.* Let $E_{n,\epsilon} = \{x : |f_n(x) - f(x)| \geqslant \epsilon\}$. Then, $\int |f_n - f| \geqslant \int_{E_{n,\epsilon}} |f_n - f| \geqslant \epsilon\mu(E_{n,\epsilon})$, and hence $\mu(E_{n,\epsilon}) \leqslant \epsilon^{-1} \int |f_n - f| \to 0$ as $n \to \infty$. $\square$

**Theorem 2.10 (Erogoff's Theorem, Almost Uniform Convergence).** *Let $\mu(X) < \infty$ and $f_1, f_2, \ldots, f$ be measurable real-valued functions on $X$ such that $f_n \to f$ almost everywhere. Then, for every $\epsilon > 0$, there exists $E \subseteq X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on $E^{\mathsf{c}}$.*

*Remark* 4. $\{f_n\}$ is said to converge to $f$ almost uniformly if for every $\epsilon > 0$, there exists $E \in \mathcal{M}$ of measure $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on $E^{\mathsf{c}}$.

*Proof.* Without loss of generality, assume that $f_n \to f$ everywhere on $X$. For $k, n \in \mathbb{N}$, let

$$E_n(k) = \cup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| \geqslant {}^1/_k\}.$$

Then, for fixed $k$, $E_n(k)$ decreases as $n$ increases and $\cap_{n=1}^{\infty} E_n(k) = \emptyset$. Since $\mu(X)\infty$, we conclude that $\lim_{n\to\infty} \mu(E_n(k)) = 0$. Given $\epsilon > 0$ and $k \in \mathbb{N}$, choose $n_k$ large enough that $\mu(E_n(k)) < \epsilon 2^{-k}$ and let $E = \cup_{k=1}^{\infty} E_{n_k}(k)$. Then, $\mu(E) < \epsilon$ and we have $|f_n(x) - f(x)| < {}^1/_k$ for $n > n_k$ and $x \notin E$. Thus, $f_n \to f$ uniformly on $E^{\mathsf{c}}$. $\square$