Lecture 12: Detection of Signals With Random Parameters

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In the previous lecture the detection of deterministic signals was dealt with. In this lecture, the focus will be on deciding between signals that are known except for a set of unknown random parameters. For this situation the hypothesis testing problem can be conveniently written as,

$$\begin{aligned} &H_0 : Y_k = N_k + s_{0k}(\Theta), \quad k = 1, 2, 3, ...n \\ \text{versus} & \\ &H_1 : Y_k = N_k + s_{1k}(\Theta), \quad k = 1, 2, 3, ...n \end{aligned} \tag{1}$$

where Θ is an unknown parameter taking values from a parameter set Λ. Density of Θ is ω_j under the hypothesis H_j, $j \in \{0, 1\}$. Moreover, $\underline{N}$ is random noise which is independent of the signal and hence independent of Θ. The likelihood ratio for (1) is given by,

$$L(\underline{y}) = \frac{\mathbb{E}_1\{p_{\underline{N}}(\underline{y} - \underline{s}_1(\Theta))\}}{\mathbb{E}_0\{p_{\underline{N}}(\underline{y} - \underline{s}_0(\Theta))\}} \quad \forall \underline{y} \in \mathbb{R}^n.$$

Here $\mathbb{E}_j$ denotes the expectation under density ω_j.

$$L(\underline{y}) = \frac{\int_\Lambda p_{\underline{N}}(\underline{y} - \underline{s}_1(\theta))\omega_1(\theta)\mu(d\theta)}{\int_\Lambda p_{\underline{N}}(\underline{y} - \underline{s}_0(\theta))\omega_0(\theta)\mu(d\theta)}. \tag{2}$$

Without loss of generality, we can assume that $\underline{s_0} \equiv 0$ and $\underline{s_1} \triangleq \underline{s}$. Applying this in (2), we have

$$L(\underline{y}) = \int_\Lambda \frac{p_{\underline{N}}(\underline{y} - \underline{s}(\theta))}{p_{\underline{N}}(\underline{y})}\omega(\theta)\mu(d\theta), \tag{3}$$

$$= \int_\Lambda L_\theta(\underline{y})\omega(\theta)\mu(d\theta), \tag{4}$$

where $L_\theta(\underline{y})$ is likelihood ratio conditioned on $\Theta = \theta$.

1 Non-coherent Detection of a Modulated Sinusoidal Carrier

Now let us consider the example of non-coherent detection of a modulated sinusoidal carrier,

$$H_0 : s_0(\theta) = \underline{0},$$
$$H_1 : s_1(\theta) = \underline{s}(\theta) = (s_k(\theta))_{k=1}^n,$$

where $s_k(\theta)$ is given by,

$$s_k(\theta) = a_k \sin[(k-1)\omega_c T_s + \theta], \qquad k = 1, 2, 3, ...n$$

where $a_1, a_2, ..., a_n$ are a known amplitude sequence and Θ is random phase uniformly distributed in the interval [0,2π], and where ω_c and T_s are a known carrier frequency and sampling interval respectively with relationship $n(\omega_c T_s) = m(2\pi)$ for some integer m. Choice of m is in such a way that the number of samples taken per cycle of the sinusoid is an integer greater than 1 (i.e, m divides n). These signals provide a model for a digital signalling scheme in which a "zero" is transmitted by sending nothing and "one" is transmitted by sending a signal modulated by a sinusoidal carrier of frequency ω_c (often called On-Off Keying).

Assuming i.i.d. $\mathcal{N}(0, \sigma^2)$ noise, the likelihood ratio in (4) can be rewritten as,

$$L(\underline{y}) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(\frac{1}{\sigma^2}\big(\sum_{k=1}^n y_k s_k(\theta) - \frac{1}{2}\sum_{k=1}^n s_k^2(\theta)\big)\right) d\theta. \tag{5}$$

First term in parenthesis in the exponent in (5) can be written as:

$$\sum_{k=1}^n y_k s_k(\theta) = \sum_{k=1}^n a_k y_k [\sin[(k-1)\omega_c T_s]\cos\theta + \cos[(k-1)\omega_c T_s]\sin\theta], \tag{6}$$
$$= y_c \sin\theta + y_s \cos\theta. \tag{7}$$

Equation in (7) is obtained by using the identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$. Moreover y_c and y_s are defined as follows:

$$y_c \triangleq \sum_{k=1}^n a_k y_k \cos[(k-1)\omega_c T_s],$$

$$y_s \triangleq \sum_{k=1}^n a_k y_k \sin[(k-1)\omega_c T_s].$$

Now we consider the second term in parenthesis in the exponent in (5)

$$-\frac{1}{2}\sum_{k=1}^{n} s_k^2(\theta) = -\frac{1}{2}\sum_{k=1}^{n} a_k^2\left[\frac{1}{2} - \frac{1}{2}\cos[2(k-1)\omega_c T_s + 2\theta]\right], \tag{8}$$
$$= -\frac{1}{4}\sum_{k=1}^{n} a_k^2 + \frac{1}{4}\sum_{k=1}^{n} a_k^2 \cos[2(k-1)\omega_c T_s + 2\theta].$$

The equation in (8) above is obtained using the identity $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$. For most of the situations in practice, the second term in (8) is zero or approximately zero for all values of θ. For example, if the signal sequence $a_1, a_2, ..., a_n$ is a constant times the sequence of ± 1's or if $a_1, a_2, ..., a_n$ has a raised cosine shape, then the above second term is identically zero. In other cases of interest, the sequence $a_1^2, a_2^2, ..., a_n^2$ is slowly varying as compared to twice the carrier frequency. So the above second term amounts to low pass filtering of a high frequency signal, which yields a negligible output. Thus $L(\underline{y})$ becomes

$$L(\underline{y}) = \exp\left(-\frac{n\bar{a^2}}{4\sigma^2}\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-\frac{1}{\sigma^2}(y_c\sin\theta + y_s\cos\theta)\right]d\theta, \tag{9}$$

where,

$$\bar{a^2} = \frac{1}{n}\sum_{k=1}^{n} a_k^2. \tag{10}$$

The integral term in (9) can be written as

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-\frac{1}{\sigma^2}(y_c\sin\theta + y_s\cos\theta)\right]d\theta = I_0\left(\frac{r}{\sigma^2}\right), \tag{11}$$

where $r = \sqrt{y_c^2 + y_s^2}$ and I_0 is the modified Bessel function of first kind and order zero. We know that $I_0(x)$ is monotonic in x. Hence the optimum tests in this case can be given as

$$\tilde{\delta}_0(\underline{y}) = \begin{cases} 1, & > \\ \gamma, & \text{if } r \quad = \sigma^2 I_0^{-1}\left(\tau e^{\frac{n\bar{a^2}}{4\sigma^2}}\right) \\ 0. & < \end{cases} \tag{12}$$

The structure of the detector is as shown in the Figure 1. The observed signal $y_1, y_2, ..., y_n$ is split into two channels one being in-phase channel and the other quadrature channel. Each channel correlates the resulting product with the amplitude sequence $a_1, ..., a_n$. The channel outputs are then combined to give r, which is compared to a threshold. This structure is also called envelope detector.

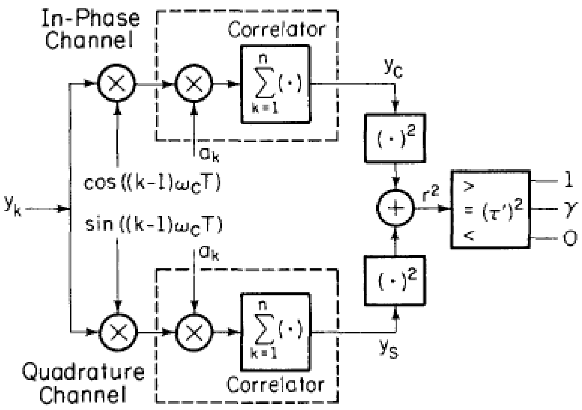

Figure 1: Optimum system for non-coherent detection of a modulated sinusoid in i.i.d. Gaussian noise
(Source: H. Vincent Poor, An Introduction to Signal Detection and Estimation(Second Edition), Figure III.B.10)

2 Performance Analysis of Optimum System for non-coherent detection

To analyze the performance of the detector, we need to find $\mathcal{P}_j(\Gamma_1)$, j $\in \{0, 1\}$ which is same as $\mathcal{P}_j(R > \tau)$, where $R = Y_c^2 + Y_s^2$ and,

$$Y_c \triangleq \sum_{k=1}^{n} a_k Y_k \cos[(k-1)\omega_c T_s].$$

$$Y_s \triangleq \sum_{k=1}^{n} a_k Y_k \sin[(k-1)\omega_c T_s].$$

The desired probabilities can be found from the joint probability density function of Y_c and Y_s under two hypothesis.

Under Hypothesis H_0,

$$H_0 : \underline{Y} \sim \mathcal{N}(0, \sigma^2 \underline{I}),$$

Y_c and Y_s are jointly Gaussian. We can specify the joint density of (Y_c, Y_s) under H_0 by finding the means and variances of Y_c and Y_s, and the correlation coefficient between Y_c and Y_s. Clearly,

$$\mathbb{E}\{Y_c|H_0\} = \mathbb{E}\{Y_s|H_0\} = 0,$$

$$
\begin{aligned}
Var\big[Y_c|H_0\big] &= \sum_{k=1}^{n}\sum_{l=1}^{n} a_k a_l \mathbb{E}\big\{N_k, N_l\big\}\cos[(k-1)\omega_c T_s]\cos[(l-1)\omega_c T_s],\\
&= \sum_{k=1}^{n} a_k^2\sigma^2\cos^2[(k-1)\omega_c T_s],\\
&= \sum_{k=1}^{n} \frac{a_k^2\sigma^2}{2},\\
&= \frac{\sigma^2 n\bar{a}^2}{2},\\
&= Var\big[Y_s|H_0\big].
\end{aligned}
$$

Cross covariance is given by,

$$
\begin{aligned}
Cov\big(Y_c, Y_s|H_0\big) &= \mathbb{E}\big\{Y_c, Y_s|H_0\big\},\\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} a_k a_l \mathbb{E}\big\{N_k, N_l\big\}\cos[(k-1)\omega_c T_s]\sin[(l-1)\omega_c T_s],\\
&= \sum_{k=1}^{n} a_k^2\sigma^2\cos[(k-1)\omega_c T_s]\sin[(k-1)\omega_c T_s],\\
&= 0.
\end{aligned}
$$

Since cross covariance of Y_c and Y_s is zero, Y_c and Y_s are uncorrelated. This implies Y_c and Y_s are independent (For Gaussian random variables zero cross correlation implies independence).

The false alarm probability thus becomes,

$$
\begin{aligned}
\mathcal{P}_{\mathcal{F}}(\tilde{\delta}_{NP}) = \mathcal{P}_0(\Gamma_1) &= \mathcal{P}_0(Y_c^2 + Y_s^2 > (\tau')^2),\\
&= \iint\limits_{(y_c^2+y_s^2>(\tau')^2)} \frac{1}{\pi n\sigma^2\bar{a}^2}\exp\left(\frac{-(y_c^2+y_s^2)}{n\sigma^2\bar{a}^2}\right)dy_c dy_s,\\
&= \int\limits_{\psi=0}^{2\pi}\int\limits_{r=\tau'}^{\infty} \frac{r}{\pi n\sigma^2\bar{a}^2}\exp\left(\frac{-r^2}{n\sigma^2\bar{a}^2}\right)dr d\psi,\\
&= \exp\left(\frac{-(\tau')^2}{n\sigma^2\bar{a}^2}\right).
\end{aligned}
$$

To find detection probability $\mathcal{P}_1(\Gamma_1)$, we need to find the joint density of (Y_c, Y_s) under H_1. We know that given $\Theta = \theta$, $\underline{Y}$ has a conditional $\mathcal{N}(\underline{s}(\theta), \sigma^2 I)$ distribution under H_1. So, given $\Theta = \theta$, Y_c and Y_s are conditionally jointly Gaussian.

$$\begin{aligned}
\mathbb{E}\{Y_c|H_1, \Theta = \theta\} &= \sum_{k=1}^{n} a_k \mathbb{E}\{Y_k|H_1, \Theta = \theta\} \cos[(k-1)\omega_c T_s], \\
&= \sum_{k=1}^{n} a_k^2 \sin[(k-1)\omega_c T_s + \theta] \cos[(k-1)\omega_c T_s], \\
&= \sum_{k=1}^{n} a_k^2 \frac{\sin\theta}{2}, \\
&= \frac{n\bar{a^2}}{2} \sin\theta.
\end{aligned}$$

Similarly,

$$\mathbb{E}\{Y_s|H_1, \Theta = \theta\} = \frac{n\bar{a^2}}{2} \cos\theta.$$

With θ fixed, the variances and covariance of Y_c and Y_s under H_1 is same as their corresponding variances and covariance under H_0 since the only change in $\underline{Y}$ is a shift in mean. So,

$$Var\big[Y_c|H_1, \Theta = \theta\big] = Var\big[Y_s|H_1, \Theta = \theta\big] = Var\big[Y_c|H_0\big] = \frac{\sigma^2 n\bar{a^2}}{2}.$$

and $Cov\big(Y_c, Y_s|H_1, \Theta = \theta\big) = 0$. Hence the joint unconditioned pdf of Y_c and Y_s under H_1 is obtained by averaging the conditional density over θ:

$$\begin{aligned}
f_{Y_c,Y_s}\big(y_c, y_s|H_1\big) &= \frac{1}{2\pi} \int_0^{2\pi} f_{Y_c,Y_s}\big(y_c, y_s|H_1, \Theta = \theta\big) d\theta, \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\pi n \sigma^2 \bar{a^2}} \exp\left(\frac{-q(y_c, y_s; n\bar{a^2}/2, \theta)}{n\sigma^2 \bar{a^2}}\right) d\theta,
\end{aligned}$$

where $q(a, b; c, \theta) = (a - c\cos\theta)^2 + (b - c\sin\theta)^2$.

Detection Probability thus becomes,

$$\begin{aligned}
\mathcal{P}_{\mathcal{D}}(\tilde{\delta}_0) &= \mathcal{P}_1(\Gamma_1) = \mathcal{P}_0(Y_c^2 + Y_s^2 > (\tau')^2), \\
&= \int\limits_{\psi=0}^{2\pi} \int\limits_{r=\tau'}^{\infty} \left(\frac{e^{(-n\bar{a^2}/4\sigma^2)}}{\pi n \sigma^2 \bar{a^2}}\right) r \exp\left(\frac{-r^2}{n\sigma^2 \bar{a^2}}\right) \mathcal{I}_0\left(\frac{r}{\sigma^2}\right) dr d\psi, \\
&= \int\limits_{x=\tau'}^{\infty} x \exp\left(-\frac{(x^2 + b^2)}{2}\right) \mathcal{I}_0(bx) dx, \\
&= \mathcal{Q}(b, \tau_0).
\end{aligned}$$

where $b^2 = n\bar{a^2}/2\sigma^2$, $\tau_0 = \tau'/\sigma^2 b$, $x = r/\sigma^2 b$, and $\mathcal{Q}(.,.)$ is "Marcum's Q-function". We set the threshold τ' for α-level Neyman-Pearson detection in this problem as follows.

$$\mathcal{P}_0(\Gamma_1) = \mathcal{P}_{\mathcal{F}}(\tilde{\delta}_{NP}) = \alpha,$$
$$\Rightarrow \exp\left(\frac{-(\tau')^2}{n\sigma^2\bar{a^2}}\right) = \alpha,$$
$$\Rightarrow \tau' = \sqrt{n\sigma^2\bar{a^2}\log\left(\frac{1}{\alpha}\right)}.$$

which gives Receiver operating characteristics(ROCs) as,

$$P_D(\tilde{\delta}) = \mathcal{Q}\left(b, \sqrt{2\log\left(\frac{1}{\alpha}\right)}\right).$$

Receiver operating characteristics look very similar to those for the coherent problem (Figure 2). We see that the performance of Neyman-Pearson detection

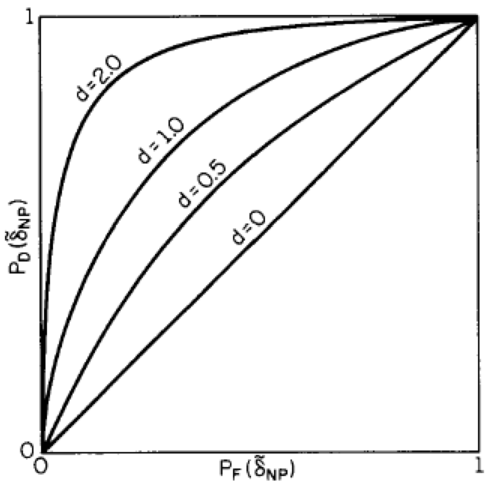

Figure 2: Receiver operating characteristics (R.O.Cs) for Neyman-Pearson non coherent detection with i.i.d. Gaussian noise.
(Source: H. Vincent Poor, An Introduction to Signal Detection and Estimation(Second Edition), Figure II.D.4)

depends only on the parameter b. The average signal energy is given by

$$\mathbb{E}\left\{\frac{1}{n}\sum_{k=1}^{n} s_k^2(\Theta)\right\} = \frac{1}{n}\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=1}^{n} a_k^2 \sin^2[(k-1)\omega_c T_s + \theta]d\theta = \frac{\bar{a^2}}{2}. \tag{13}$$

Here b^2 has a signal-to-noise ratio interpretation similar to d^2 in coherent detection problem. If θ is known, to detect the same signal coherently, the value of d^2 would be

$$d^2 = \frac{1}{\sigma^2}\sum_{k=1}^{n} s_k^2(\theta) = \frac{1}{\sigma^2}\sum_{k=1}^{n} a_k^2 \sin^2[(k-1)\omega_c T_s + \theta] = \frac{n\bar{a}^2}{2\sigma^2} = b^2. \tag{14}$$

Thus these signal-to-noise ratios are same. However, the performance for fixed α is different for the two systems. For typical SNR and α values, we have

$$\mathcal{Q}\left(b, \sqrt{2log\frac{1}{\alpha}}\right) = 1 - \Phi(\Phi^{-1}(1-\alpha) - d), \tag{15}$$

and the equality holds when $b \approx d + 0.4$. This means that we need a slightly higher SNR to get the same performance as that of coherent technique.