Lecture 02: Poisson Process

1 Simple point processes

Definition 1.1. A stochastic process $\{N(t), t \geqslant 0\}$ is a **point process** if

1. $N(0) = 0$, and
2. for each $\omega \in \Omega$, the map $t \mapsto N(t)$ is non-decreasing, integer valued, and right continuous.

Definition 1.2. A **simple point process** is a point process of jump size 1.

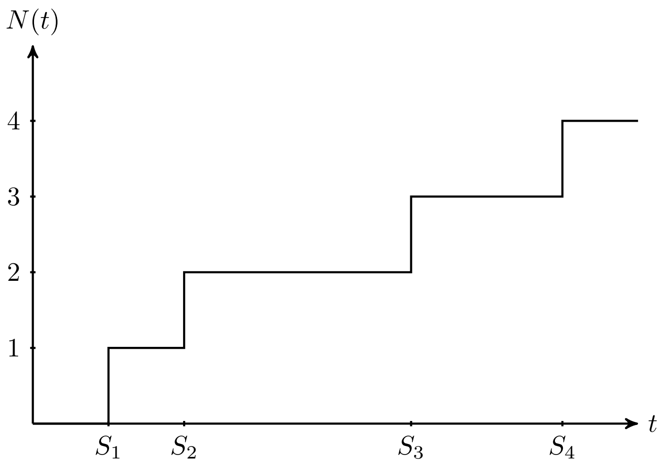

Figure 1: Sample path of a simple point process.

Definition 1.3. We can define a random variable S_n as the time of n^{th} discontinuity, written

$$S_n = \inf\{t \geq 0 : N(t) = n\}, n \in \mathbb{N}, \qquad S_0 = 0.$$

The points of discontinuity corresponds to the arrival instants of the point process.

Lemma 1.4. *Simple point process* $\{N(t), t \geqslant 0\}$ *and arrival process* $\{S_n : n \in \mathbb{N}\}$ *are inverse processes. That is,*

$$\{S_n \leqslant t\} = \{N(t) \geqslant n\}.$$

Proof. Let $\omega \in \{S_n \leqslant t\}$, then $N(S_n) = n$. Since N is a non-decreasing process, we have $N(t) \geq N(S_n) = n$. Conversely, let $\omega \in \{N(t) \geqslant n\}$, then it follows from definition that $S_n \leq t$. □

Corollary 1.5. *The following identity is true.*

$$\{S_n \leqslant t, S_{n+1} > t\} = \{N(t) = n\}.$$

Lemma 1.6. *Let $F_n(x)$ be the distribution function for S_n, then*

$$P_n(t) \triangleq \Pr\{N(t) = n\} = F_n(t) - F_{n+1}(t).$$

Proof. It suffices to observe that following is a union of disjoint events,

$$\{S_n \leqslant t, S_{n+1} > t\} \cup \{S_n \leqslant t, S_{n+1} \leqslant t\} = \{S_n \leqslant t\}.$$

□

Definition 1.7. The inter arrival time between $(n-1)^{th}$ and n^{th} arrival is denoted by X_n and written as

$$X_n = S_n - S_{n-1}.$$

Remark 1.8. For a simple point process, we have

$$\Pr\{X_n = 0\} = \Pr\{X_n \leqslant 0\} = 0.$$

Definition 1.9. A point process $\{N(t), t \geqslant 0\}$ is called **stationary increment point process**, if for any collection of $0 < t_1 < t_2, \ldots, < t_n$, the joint distribution of $(N(t_n) - N(t_{n-1}), N(t_{n-1}) - N(t_{n-2}), ..., N(t_1))$ is identical to the joint distribution of $(N(t_n + t) - N(t_{n-1} + t), ..., N(t_1 + t))$, $\forall t \geqslant 0$.

Definition 1.10. A point process $\{N(t), t \geqslant 0\}$ is called **stationary independent increment process**, if it has stationary increments and the increments are independent random variables.

Lemma 1.11. *Sequence of inter-arrival times $\{X_n : n \in \mathbb{N}\}$ of a simple stationary independent increment process $\{N(t), t \geqslant 0\}$ consists of iid random variables.*

Proof. It suffices to show that X_n is independent of S_{n-1} to show that all inter-arrival times are independent. We see that

$$\begin{aligned}\Pr\{S_n \leqslant x, S_{n+1} - S_n > y\} &= \int_{x \le t} \Pr\{N(y+t) - N(t) = 0 | S_n = t\} dF_n(t) \\ &= \int_{x \le t} \Pr\{N(y+t) - N(t) = 0 | N(t) = n\} dF_n(t) = \Pr\{X_n > y\} F_n(x).\end{aligned}$$

To show that each inter-arrival time is identically distributed, we observe that

$$\begin{aligned}\Pr\{S_n - S_{n-1} > x\} &= \Pr\{N(x + S_{n-1}) - N(S_{n-1}) > 0\} \\ &= \int_{t>0} \Pr\{N(x) = 0\} dF_{n-1}(t) = \Pr\{N(x) = 0\}.\end{aligned}$$

□

2 Poisson process

Lemma 2.1. *A unique non-negative right continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying align*

$$f(t+s) = f(t)f(s), \text{ for all } t, s \in \mathbb{R}$$

is $f(t) = e^{\theta t}$, where $\theta = \log f(1)$.

Proof. Clearly, we have $f(0) = f^2(0)$. Since f is non-negative, it means $f(0) = 1$. By definition of θ and induction for $m, n \in \mathbb{Z}^+$, we see that

$$f(m) = f(1)^m = e^{\theta m}, \qquad e^{\theta} = f(1) = f(1/n)^n.$$

Let $q \in \mathbb{Q}$, then it can be written as $m/n, n \neq 0$ for some $m, n \in \mathbb{Z}^+$. Hence, it is clear that for all $q \in \mathbb{Q}^+$, we have $f(q) = e^{\theta q}$. either unity or zero. Note, that f is a right continuous function and is non-negative. Now, we can show that f is exponential for any real positive t by taking a sequence of rational numbers $\{t_n\}$ decreasing to t. From right continuity of g, we obtain

$$g(t) = \lim_{t_n \downarrow t} g(t_n) = \lim_{t_n \downarrow t} e^{\beta t_n} = e^{\beta t}.$$

□

Definition 2.2. A random variable X with continuous support on $\mathbb{R}_+$, is called **memoryless** if for all $t, s \in \mathbb{R}_+$, we have

$$\Pr\{X > s\} = \Pr\{X > t + s | X > t\}.$$

Proposition 2.3. *The unique memoryless distribution function with continuous support on $\mathbb{R}_+$ is the exponential distribution.*

Proof. Let X be a random variable with a distribution function $F : \mathbb{R}_+ \to [0, 1]$ with the memoryless property. Let $g(t) \triangleq 1 - F(t)$. It follows from the memoryless property of F, that

$$g(t+s) = g(t)g(s).$$

Since $g(x) = \Pr\{X > x\}$ is non-increasing with $x \in \mathbb{R}_+$, we have $g(x) = e^{\theta x}$, where $\theta < 0$. □

Definition 2.4. A simple point process $\{N(t), \ t \geqslant 0\}$ is called a **Poisson process** with a finite positive rate λ, if inter-arrival times $\{X_n : n \in \mathbb{N}\}$ are iid random variables with an exponential distribution of rate λ. That is, it has a distribution function F, such that

$$F(x) = \Pr\{X_1 \leqslant x\} = \begin{cases} 1 - e^{-\lambda x}, & x \geqslant 0 \\ 0, & \text{else.} \end{cases}$$

Theorem 2.5. *A simple stationary independent increment process is a Poisson process with parameter λ when*

$$\lim_{t \to 0} \frac{\Pr\{N(t) = 1\}}{t} = \lambda, \qquad \lim_{t \to 0} \frac{\Pr\{N(t) \geq 2\}}{t} = 0.$$

Proof. It suffices to show that first inter-arrival times X_1 is exponentially distributed with parameter λ. Notice that

$$P_0(t+s) = \Pr\{N(t+s) - N(t) = 0, N(t) = 0\} = P_0(t)P_0(s).$$

Using the conditions in the theorem, the result follows. □

2.1 Distribution functions

Lemma 2.6. *Moment generating function of arrival times S_n is*

$$\mathbb{E}[e^{\theta S_n}] = \begin{cases} \frac{\lambda^n}{(\lambda-\theta)^n}, & \theta < \lambda \\ \infty, & \theta \geqslant \lambda. \end{cases}$$

Distribution function of S_n is given by

$$.$$

Proof. Since $S_n = \sum_{k=1}^n X_k$, where X_k are iid, the moment generating function $\mathbb{E}[e^{\theta S_n}]$ of S_n is

$$\mathbb{E}[e^{\theta S_n}] = \left(\mathbb{E}[e^{\theta X_1}]\right)^n.$$

Since each X_k is iid exponential with rate λ, it is easy to see that moment generating function of inter-arrival time X_1 is

$$\mathbb{E}[e^{\theta X_1}] = \begin{cases} \frac{\lambda^n}{(\lambda-\theta)^n}, & \theta < \lambda \\ \infty, & \theta \geqslant \lambda. \end{cases}$$

□

Theorem 2.7. *Density function of S_n is Gamma distributed with parameters n and λ. That is,*

$$f_n(s) = \frac{\lambda(\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}.$$

Proof. Notice that X_i's are iid and $S_1 = X_1$. In addition, we know that $S_n = X_n + S_{n-1}$. Since, X_n is independent of S_{n-1}, we know that distribution of S_n would be convolution of distribution of S_{n-1} and X_1. Since X_n and S_1 have identical distribution, we have $f_n = f_{n-1} * f_1$. The result follows from straightforward induction. □

Theorem 2.8. *For each $t > 0$, the distribution of Poisson process $N(t)$ with parameter λ is given by*

$$\Pr\{N(t) = n)\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

Further, $E[N(t)] = \lambda t$, explaining the rate parameter λ for Poisson process.

Proof. Result follows from density of S_n and recognizing that

$$P_n(t) = F_n(t) - F_{n+1}(t).$$

□

Corollary 2.9. *Distribution of arrival times S_n is*

$$F_n(t) = \sum_{j \geq n} e^{-\lambda t}\frac{(\lambda t)^j}{j!}.$$

Further, $\sum_{n\in\mathbb{N}_0} F_n(x) = 1 + \lambda t$.

Proof. Result follows from distribution of $P_n(t)$ and recognizing that $F_n(t) = \sum_{j \geq n} P_j(t)$. Further, we notice that

$$\sum_{n \in \mathbb{N}_0} F_n(t) = \sum_{n \in \mathbb{N}_0} \sum_{j \geq n} P_j(t) = \sum_{i \in \mathbb{N}_0} \sum_{n=0}^{j} P_j(t) = \sum_{i \in \mathbb{N}_0} (j+1) P_j(t)$$
$$= 1 + E[N(t)] = 1 + \lambda t.$$

□

Remark 2.10. A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant.

Lemma 2.11. *For any finite time $t > 0$, a Poisson process is finite almost surely.*

Proof. By strong law of large numbers, we have

$$\lim_{n \to \infty} \frac{S_n}{n} = E[X_1] = \frac{1}{\lambda} \quad \text{a.s.}$$

Fix $t > 0$ and let $M = \{\omega \in \Omega : N(t)(\omega) = \infty\}$ be a subset of the sample space. Let $\omega \in M$, then $S_n(\omega) \leqslant t$ for all $n \in \mathbb{N}$. This implies $\limsup_n \frac{S_n}{n} = 0$ and $\omega \notin \{\lim_n \frac{S_n}{n} = \frac{1}{\lambda}\}$. Hence, the probability measure for set M is zero. □

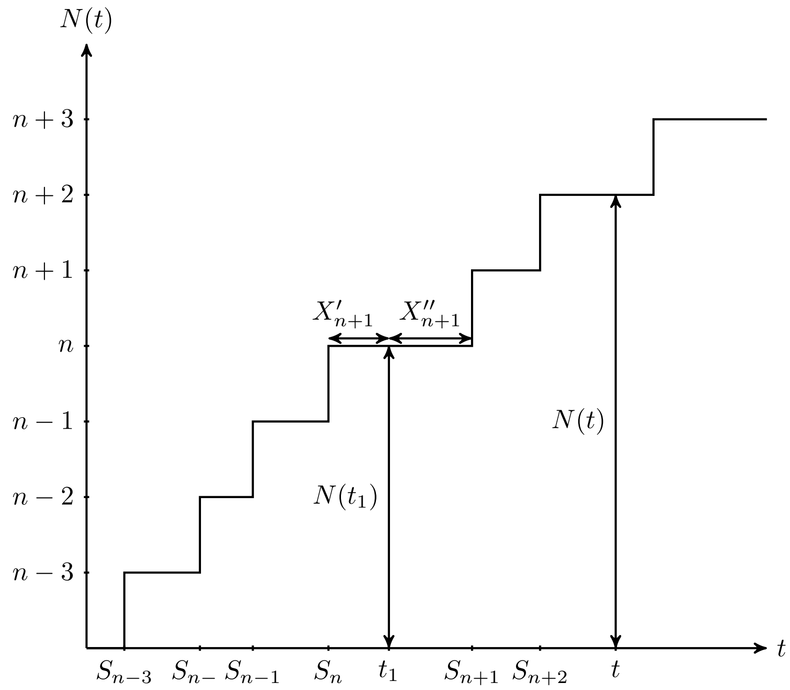

Figure 2: Stationary independent increment property of Poisson process.

Proposition 2.12. *A Poisson process $\{N(t), t \geqslant 0\}$ is simple point process with stationary independent increments.*

Proof. It is clear that Poisson process is a simple point process. To show that $N(t)$ has stationary independent increment property, it suffices to show that $N_t - N(t_1) \perp N(t_1)$ and $N(t) - N(t_1) \sim N(t - t_1)$. This follows from the fact that we can use induction to show stationary independent increment property for for any finite disjoint time-intervals.

Let arrival time-instants $\{S_n : n \in \mathbb{N}_0\}$ and inter-arrival times $\{X_n : n \in \mathbb{N}\}$ be defined as before. Given any time t_1, we can define the following variables

$$X'_{N(t_1)+1} = t_1 - S_{N(t_1)}, \qquad X''_{N(t_1)+1} = S_{N(t_1)+1} - t_1.$$

It is clear that t_1 partitions $X_{N(t_1)+1}$ in two parts such that $X_{N(t_1)+1} = X^{'}_{N(t_1)} + 1 + X^{''}_{N(t_1)+1}$ as seen in Figure **??** for the case when $N(t_1) = n$. We look at joint distribution of $X'_{N(t_1)+1}, X''_{N(t_1)+1}$ and notice that

$$\begin{aligned}\{X'_{N(t_1)+1} > x, X''_{N(t_1)+1} > y\} &= \bigcup_{n \in \mathbb{N}_0} \{S_n < t_1 - x, S_{n+1} > t_1 + y, N(t_1) = n\} \\ &= \bigcup_{n \in \mathbb{N}_0} \{S_n < t_1 - x, S_{n+1} > t_1 + y\}.\end{aligned}$$

From the fact that inter-arrival times are iid exponentially distributed with rate λ, we conclude that

$$\begin{aligned}\Pr\{X'_{N(t_1)} > x, X''_{N(t_1)+1} > y\} &= \sum_{n \in \mathbb{N}_0} \int_{u=0}^{t_1 - x} \Pr\{X_{n+1} > t_1 + y + u\} dF_n(u), \\ &= \int_{u=0}^{t_1-x} (1 - F_1(t_1 + y + u)) \sum_{n \in \mathbb{N}_0} dF_n(u) = \int_{u=0}^{t_1-x} e^{-\lambda(t_1+y+u)} \lambda du, \\ &= (1 - F_1(y))(F_1(t_1) - F_1(2t_1 - x)).\end{aligned}$$

Therefore, $X^{''}_{N(t_1)+1}$ is independent of $X^{'}_{N(t_1)+1}$ and has same distribution as X_{n+1}. The memoryless property of exponential distribution is crucially used. Further, we see that independent increment holds only if inter-arrival time is exponential. Therefore,

$$\begin{aligned}\{N(t_1) = n\} &\iff \{S_n = t_1 + X'_{n+1}\}, \\ \{N(t) - N(t_1) \geqslant m\} &\iff \{X''_{n+1} + \sum_{i=n+2}^{n+m} X_i \leqslant t - t_1\}.\end{aligned}$$

Since, $\{X_i : i \geqslant n+2\} \cup \{X^{''}_{n+1}\}$ are independent of $\{X_i : i \leqslant n\} \cup X^{'}_{n+1}$, we have $N(t) - N(t_1) \perp N(t_1)$. Further, since $X^{''}_{n+1}$ has same distribution as X_{n+1}, we get $N(t) - N(t_1) \sim N(t - t_1)$. By induction we can extend this result to $(N(t_n) - N(t_{n-1}), ..., N(t_1))$. ☐

Proposition 2.13. *Let $t_0 = 0$, and $\{t_i : 1 \leq i \leq k\}$ be an increasing sequence. A stationary independent increment point process $\{N(t),\ t \geqslant 0\}$, such that $N(0) = 0$ is Poisson process iff*

$$\Pr\{\bigcap_{i=1}^{k} \{N(t_i) - N(t_{i-1}) = n_i\}\} = \prod_{i=1}^{k} \frac{(\lambda(t_i - t_{i-1}))^{n_i}}{n_i!} e^{-\lambda(t_i - t_{i-1})}.$$