Lecture 03: Properties of Poisson Process

1 Conditional Distribution of Arrivals

Proposition 1.1. *Let $\{N(t), t \geqslant 0\}$ be a Poisson process with $\{A_i \subseteq \mathbb{R}_+ : i \in [n]\}$ a set of finite disjoint intervals with $B = \cup_{i\in[n]} A_i$, and $\{k_i \in \mathbb{N} : i \in [n]\}$ and $k = \sum_{i\in[n]} k_i$. Then, we have*

$$\Pr \bigcap_{i\in[n]} \{N_{A_i} = k_i | N(B) = k\} = k! \prod_{i\in[n]} \frac{1}{k_i!} \left(\frac{|A_i|}{|B|}\right)^{k_i}.$$

Proof. It follows from the stationary independent increment property of Poisson processes that

$$\Pr \bigcap_{i\in[n]} \{N_{A_i} = k_i | N(B) = k\} = \frac{\Pr \bigcap_{i\in[n]} \{N_{A_i} = k_i\}}{\Pr\{N_B = k\}} = \frac{1}{\Pr\{N_B = k\}} \prod_{i\in[n]} \Pr\{N_{A_i} = k_i\}.$$

□

Proposition 1.2. *For a Poisson process $\{N(t), t \geqslant 0\}$, distribution of first arrival instant S_1 conditioned on $\{N(t) = 1\}$ is uniform between $[0, t)$.*

Proof. If $N(t) = 1$, then we know that conditional distribution of S_1 is supported on $[0, t)$. By Proposition **??**, we see that

$$\Pr\{S_1 \leq s | N(t) = 1\} = \Pr\{N(s) = 1, N(t - s) = 0 | N(t) = 1\} 1_{\{s<t\}} = \frac{s}{t} 1_{\{s<t\}}.$$

□

Alternative proof. For any $0 \leq u < t$, we can write $\{S_1 = u, N(t) = 1\}$ as intersection of two independent events,

$$\{S_1 = u, N(t) = 1\} \iff \{S_1 = u\} \cap \{X_2 > t - u\}.$$

Therefore, integrating LHS with respect to u in interval $[0, s]$ for $s < t$, we obtain

$$\Pr\{S_1 \leq s, N(t) = 1\} = \int_0^s du \lambda \exp(-\lambda u) \exp(-\lambda(t - u)) = s\lambda \exp(-\lambda t).$$

Since $\Pr\{N(t) = 1\} = \lambda t \exp(-\lambda t)$, it follows that

$$\Pr\{S_1 \leq s | N(t) = 1\} = \begin{cases} \frac{s}{t}, & s < t \\ 0, & s \geq t. \end{cases}.$$

□

Proposition 1.3. *For a Poisson process $\{N(t), t \geqslant 0\}$, joint distribution of arrival instant $\{S_1, \dots, S_n\}$ conditioned on $\{N(t) = n\}$ is identical to joint distribution of order statistics of n iid uniformly distributed random variables between $[0, t]$.*

Proof. Let $\{s_0 = 0 < s_1 < s_2 < \dots < s_n < t\}$ be a finite sequence of non-negative increasing numbers between 0 and t. Then, by Proposition **??**, we get

$$\Pr \bigcap_{i \in [n]} \{S_i \leq s_i | N(t) = n\} = \Pr \bigcap_{i \in [n]} \{N((0, s_i]) \geq i | N(t) = n\}.$$

□

Alternative proof. Let $\{s_i \in (0, t) : i \in [n]\}$ be a sequence of increasing numbers. If we denote $s_0 = 0$, then we can write

$$\bigcap_{i=1}^{n} \{S_i = s_i\} \cap \{N(t) = n\} \iff \bigcap_{i=1}^{n} \{X_i = s_i - s_{i-1}\} \cap \{X_{n+1} > t - s_n\}.$$

Note that all the events on RHS are independent events. Therefore, it is easy to compute the joint distribution of $\{S_1, \dots, S_n\}$, as

$$\begin{aligned}
\Pr \bigcap_{i=1}^{n} \{S_i \leq s_i\} \cap \{N(t) = n\} &= \int_0^{s_1} du_1 \cdots \int_0^{s_n} du_n \prod_{i=1}^{n} \lambda \exp(-\lambda(u_i - u_{i-1}) \exp(-\lambda(t - u_n)) \\
&= \lambda^n \exp(-\lambda t) \prod_{i=1}^{n} s_i.
\end{aligned}$$

Since $\Pr\{N(t) = n\} = \exp(-\lambda t)(\lambda t)^n / n!$, it follows that

$$\Pr\{S_1 \leq s_1, \dots, S_n \leq s_n | N(t) = n\} = \begin{cases} n! \prod_{i=1}^{n} \frac{s_i}{t} & s < t \\ 0 & s \geq t. \end{cases}$$

Let $U_1, \dots, U_n$ are iid Uniform random variables in $[0, t]$. Then, the order statistics of $U_1 \dots, U_n$ has an identical joint distribution to n arrival instants conditioned on $\{N(t) = n\}$. □

2 Age and excess time

Definition 2.1. For a point process $\{N(t), t \geqslant 0\}$, we can define age process $\{A(t), t \geqslant 0\}$ and excess time process $\{Y(t), t \geqslant 0\}$ as

$$A(t) = t - S_{N(t)}, \qquad Y(t) = S_{N(t)+1} - t.$$

Proposition 2.2. *For a Poisson process with rate λ, the corresponding age and excess time are both exponentially distributed with rate λ irrespective of time t.*

Proof. Using stationary independent increment property of Poisson process, we can write complementary distribution of excess time process as

$$\begin{aligned}
\Pr\{Y(t) > y\} &= \sum_{n \in \mathbb{N}_0} \Pr\{Y(t) > y, N(t) = n\} = \sum_{n \in \mathbb{N}_0} \Pr\{N(t + y) - N(t) = 0, N(t) = n\} \\
&= \Pr\{N(y) = 0\} \sum_{n \in \mathbb{N}_0} \Pr\{N(t) = n\} = \Pr\{N(y) = 0\}.
\end{aligned}$$

Similarly, we can write complementary distribution for the age process as

$$\begin{aligned}\Pr\{A(t) \geq x\} &= \sum_{n\in\mathbb{N}_0} \Pr\{A(t) \geq x, N(t) = n\} = \sum_{n\in\mathbb{N}_0} \Pr\{N(t) - N(t-x) = 0, N(t) = n\} \\ &= \sum_{n\in\mathbb{N}_0} \Pr\{N(t-x) = n\} \Pr\{N(x) = 0\} = \Pr\{N(x) = 0\}.\end{aligned}$$

□

3 Superposition and decomposition of Poisson processes

Theorem 3.1 (Sum of Independent Poissons). *Let $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ be two independent Poisson processes with rats λ_1 and λ_2 respectively. Then, the process $N(t) = N_1(t) + N_2(t)$ is Poisson with rate $\lambda_1 + \lambda_2$.*

Proof. We need to show that $\{N(t)\}$ has stationary independent increments, and

$$\Pr\{N(t) = n\} = \exp(-(\lambda_1 + \lambda_2)t) \frac{(\lambda_1 + \lambda_2)^n t^n}{n!}.$$

For two disjoint interval (t_1, t_2) and (t_3, t_4), we can see that for both processes $N_1(t)$ and $N_2(t)$, arrivals in (t_1, t_2) and (t_3, t_4) are independent. Therefore, $N(t)$ has independent increment property. Similarly, we can argue about the stationary increment property of $\{N(t)\}$. Further, we can write

$$\{N(t) = n\} = \bigcup_{k=0}^{n} \{\{N_1(t) = k\} \cap \{N_2(t) = n-k\}\}.$$

Since $N_1(t)$ and $N_2(t)$ are independent, we can write

$$\begin{aligned}\Pr\{N(t) = n\} &= \sum_{k=0}^{n} \exp(-\lambda_1 t) \frac{(\lambda_1 t)^k}{k!} \exp(-\lambda_2 t) \frac{(\lambda_2 t)^{n-k}}{(n-k)!}, \\ &= \frac{\exp(-(\lambda_1 + \lambda_2)t)}{n!} \sum_{k=0}^{n} \binom{n}{k} (\lambda_1 t)^k (\lambda_2 t)^{n-k}.\end{aligned}$$

Result follows by recognizing that summand is just binomial expansion of $[(\lambda_1 + \lambda_2)t]^n$. □

Remark 3.2. If independence condition is removed, the statement is not true.

Theorem 3.3 (Independent Spilitting). *Let $\{N(t), t \geqslant 0\}$ be a Poisson arrival process. Each arrival can be randomly assigned to either arrival type 1 or 2, with probability p and $(1-p)$ respectively, independent of previous assignments. Arrival processes of type 1 and 2 are denoted by $N_1(t)$ and $N_2(t)$ respectively. Then, $\{N_1(t), t \geqslant 0\}$,and $\{N_2(t), t \geqslant 0\}$ are mutually independent Poisson processes with rates λp and $\lambda(1-p)$ respectively.*

Proof. To show that $N_1(t), t \geq 0$ is a Poisson process with rate λp, we show that it is stationary independent increment process with the distribution

$$\Pr\{N_1(t) = n\} = \frac{(p\lambda t)^n}{n!} e^{-\lambda p t}.$$

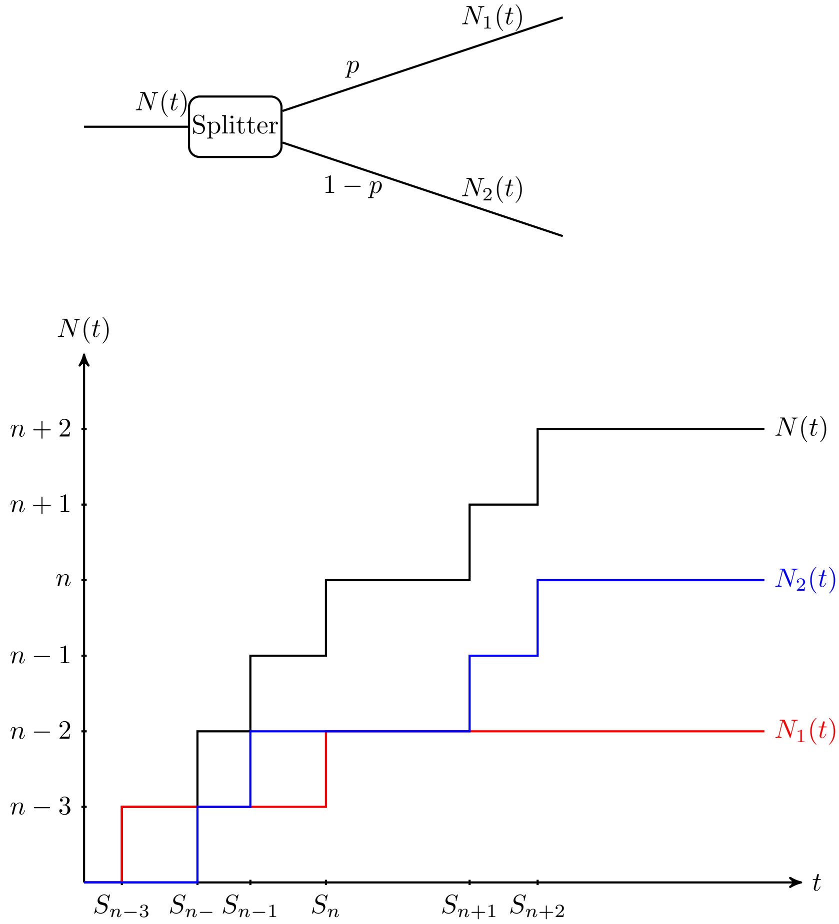

Figure 1: Splitting a Poisson process into two independent Poisson processes.

The stationary, independent increment property of the probabilistically filtered processes $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ can be understood and argued out from the example given in the figure. Notice that

$$\{N_1(t) = k\} = \bigcup_{n=k}^{\infty} \{N(t) = n, N_1(t) = k\}.$$

Further notice that conditioned on $\{N(t) = n\}$, probability of event $\{N_1(t) = k\}$ is merely probability of selecting k arrivals out of n, each with independent probability p. Therefore,

$$\begin{aligned}\Pr\{N_1(t) = k\} &= \exp(-\lambda t) \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{n!} \binom{n}{k} p^k (1-p)^{n-k}, \\ &= \exp(-\lambda t) \frac{(\lambda p t)^k}{k!} \sum_{n=k}^{\infty} \frac{(\lambda(1-p)t)^{n-k}}{(n-k)!}.\end{aligned}$$

Recognizing that infinite sum in RHS adds up $\exp(\lambda(1-p)t)$, the result follows. We can find the distribution of $N_2(t)$ by similar arguments. We will show that events $\{N_1(t) = n_1\}$ and

$\{N_2(t) = n_2\}$ are independent. To this end, we see that

$$\{N_1(t) = n_1, N_2(t) = n_2\} = \{N(t) = n_1 + n_2, N_1(t) = n_1\}.$$

Using their distribution for $N_1(t), N_2(t)$, and conditional distribution of $N_1(t)$ on $N(t)$, we can show that

$$\begin{aligned}\Pr\{N_1(t) = n_1, N_2(t) = n_2\} &= \exp(-\lambda t)\frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!}\binom{n_1+n_2}{n_1}p^{n_1}(1-p)^{n_2},\\ &= \Pr\{N_1(t) = n_1\}\Pr\{N_2(t) = n_2\}.\end{aligned}$$

In general, we need to show finite dimensional distributions factorize. That is, we need to show that for measurable sets $A_1, \dots, A_n : j \in [m]\}$, we have

$$\Pr\left(\bigcap_{i=1}^{n}\{N_1(t_i) \in A_i\}\bigcap_{j=1}^{m}\{N_2(s_j) \in B_j\}\right) = \Pr\left(\bigcap_{i=1}^{n}\{N_1(t_i) \in A_i\}\right)\Pr\left(\bigcap_{j=1}^{m}\{N_2(s_j) \in B_j\}\right).$$

□