Lecture 04: Compound Poisson Processes

1 Queueing Theory

Consider the scenario of a bus stop or a movie ticket counter. Each person arrives to the queue at a random time and has to wait another random amount of time before he is serviced. A natural question to ask is regarding the expected total waiting time of all the people in the queue. To answer this question, we first formalize the idea of a queue.

1.1 A Preliminary example

Consider a queue where the customers are arriving according to a Poisson Process $N(t)$ of rate λ. Recall that $N(t)$ is a random variable that denotes the number of arrivals till time t with S_n the time instant of n^{th} arrival. If $N(t) = n$, then the total expected waiting time is given by

$$\mathbb{E}\left[\sum_{i=1}^{N(t)}(t - S_i)\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{n}(t - S_i)\Big| N(t) = n\right]\right] \tag{1}$$

Recall that given the number of arrivals in a particular time duration, the arrivals are order statistics of uniformly distributed random variables in that time interval. Let $U_1, U_2, \dots, U_n$ be iid uniform on $[0, t]$, and let $U_{(1)}, U_{(2)}, \dots, U_{(n)}$ be their order statistics. Then

$$\mathbb{E}\left[\sum_{i=1}^{n}(t - S_i)|N(t) = n\right] = \mathbb{E}\left[\sum_{i=1}^{n}(t - U_{(i)})\Big| N(t) = n\right] = nt - \mathbb{E}\left[\sum_{i=1}^{n}U_i\Big| N(t) = n\right] = \frac{nt}{2}.$$

Substituting conditional expectation in equation 1, we obtain

$$\mathbb{E}\left[\sum_{i=1}^{N(t)}(t - S_i)\right] = \mathbb{E}\left[N(t)\right]\frac{t}{2} = \frac{\lambda t^2}{2}.$$

1.2 Notations

A queue is denoted as $G_1/G_2/K_1/K_2$ where

1. G_1 denotes the arrival distribution,
2. G_2 denotes the service distribution,
3. K_1 denotes the number of servers, and
4. K_2 denotes the size of the buffer.

Remark 1.1. Typical arrival and service distributions are taken to be independent.

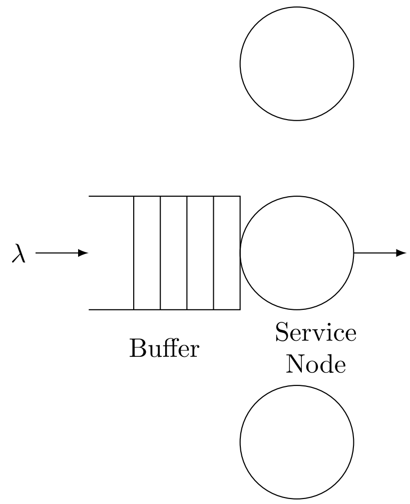

Figure 1: A general queue

Remark 1.2. Memoryless, general and deterministic distributions are denoted by M, G, and D respectively.

Remark 1.3. Number of servers and buffer sizes can be finite of infinite.

Remark 1.4. Service policy of the queue could be first in first out (FIFO), last in first out (LIFO), or Processor sharing.

1.3 $M/G/\infty$ Queue

The $M/G/\infty$ queue has a memoryless arrival distribution with infinite number of servers and a general service distribution G. Since, this queue has infinite servers, each arriving customer can enter an idle server immediately on arrival. We are interested in computing waiting time distribution of any customer in this queue. To this end, we can diving incoming customers into following two types.

1. **Type-1 customer** whose service is finished by time t.
2. **Type-2 customer** who doesn't complete the service by time t.

Probability of a customer who arrived at time s and leaves by time t is given by

$$P(s) = \Pr\{\text{service time } \leq t - s\} = G(t-s)\mathbf{1}_{\{s\leq t\}}.$$

Let $N_1(t)$ be the number of type-1 customers that arrived in duration $[0,t)$. Then

$$\mathbb{E}\left[N_1(t)\right] = p\mathbb{E}\left[N(t)\right],$$

where $p = \frac{1}{t}\int_0^t P(s)ds$.

1.4 Busy Period of $M/G/1$ Queue

The $M/G/\infty$ queue has a memoryless arrival distribution with single server and a general service distribution G. We consider FIFO service policy at the server. For a queue with single server and FIFO service policy, arriving customer enters the service only if the server is free. If the server is serving other customer, arriving customers wait in the queue.

Definition 1.5. We define **busy period** of a queue by the duration when server is busy, and denote it by B. It starts when an incoming arrival finds and idle server, and ends when there are no more customers in the system.

We are interested in characterizing the distribution of busy period B, in terms of following system parameters. We denote rate of Poisson arrival by λ, and number of customers that arrive in time duration $[0, t)$ by $N(t)$. Service times $\{Y_i : i \in \mathbb{N}\}$ of individual customers are assumed iid with distribution G and independent of the arrivals. We denote sum of k service times by $T_k = \sum_{i\in[k]} Y_i$. Since service time are iid, we can denote distribution of sum of k iid service times by G_k, the k-fold convolution of G. Without loss of generality, we start the busy period at time 0 when an arriving customer sees an idle server. We denote arrival instant of k^{th} additional customer during a busy period, by S_k.

Lemma 1.6. *Busy period is of duration t and consists of n services if and only if*

1. $S_k \leq T_k$ *for all* $k \in [n-1]$,
2. $T_n = t$,
3. $N(t) = n - 1$.

Theorem 1.7. *Distribution of busy period of an $M/G/1$ queue is given by*

$$\Pr\{B \leq t\} = \sum_{n\in\mathbb{N}} \int_0^t e^{-\lambda u} \frac{(\lambda u)^{n-1}}{n!} dG_n(u).$$

Proof. From Lemma 1.6, we can write

$$\Pr\{B \leq t, N(t) = n-1\} = \int_0^t \Pr\left\{S_k \leq T_k, k \in [n-1] \middle| N(u) = n-1, T_n = u\right\} dG_n(u)P_{n-1}(u).$$

Further, from total probability law, we know that

$$\Pr\{B \leq t\} = \sum_{n\in\mathbb{N}} \Pr\{B \leq t, N(t) = n-1\}.$$

Hence, it suffices to show that $\Pr\{S_k \leq T_k, k \in [n-1] | N(u) = n-1, T_n = u\} = 1/n$. Recall that given $N(u) = n - 1$, arrival instants $\{S_k : k \in [n-1]\}$ are order statistics of $n - 1$ iid uniform random variables $\{U_i : i \in [n-1]\}$ in $[0, u)$. Clearly, $\{u - U_i : i \in [n-1]\}$ are also iid uniform random variables in $[0, u)$, and order statistics of these random variables would be $\{S_{n-k} : k \in [n-1]\}$. Then, using Lemma 1.11 we can write

$$\Pr\left\{S_k \leq T_k, k \in [n-1] \middle| T_n = u\right\} = \Pr\left\{u - S_{n-k} \leq u - (T_n - T_{n-k}), k \in [n-1] \middle| T_n = u\right\} = \frac{1}{n}.$$

$\square$

Lemma 1.8. *Let $\{Y_i \geq 0, i \in [n]\}$ be iid random variables. Then, for all $A \subseteq [n]$, we have*

$$\mathbb{E}\left[\sum_{i\in A} Y_i \middle| \sum_{i\in[n]} Y_i = y\right] = \frac{|A|y}{n}.$$

Proof. Since Y_i's are iid, we notice that

$$y = \mathbb{E}\left[\sum_{i\in[n]} Y_i \Big| \sum_{i\in[n]} Y_i = y\right] = n\mathbb{E}\left[Y_i \Big| \sum_{i\in[n]} Y_i = y\right].$$

Hence, we can conclude the result. $\square$

Lemma 1.9. *Let $\{U_i, i \in [n]\}$ be iid random variables in $[0,t)$ and $\{U_{(i)}, i \in [n]\}$ be their order statistics. Then conditioned on $U_{(n)} = u$, $\{U_i, i \in [n-1]\}$ are the order statistics of $n-1$ uniform random variables in $[0,u)$.*

Proof. For order statistics of n uniform random variables in $[0,t)$ and for $u_1 \leq \ldots \leq u_n$, we get

$$\Pr\{U_{(1)} \leq u_1, \ldots, U_{(n)} \leq u_n\} = n! \prod_{i=1}^{n} \frac{u_i}{t}$$

Further, distribution of $U_{(n)} = \max_{i\in[n]} U_i$ is given by

$$\Pr\{U_{(n)} \leq u\} = \left(\frac{u}{t}\right)^n.$$

Combining these two results, we get

$$\Pr\{U_{(1)} \leq u_1, \ldots, U_{(n-1)} \leq u_{n-1} | U_n = u\} = (n-1)! \prod_{i=1}^{n-1} \frac{u_i}{u}.$$

$\square$

Lemma 1.10. *Let $\tau_1, \tau_2, \ldots, \tau_n$ denote the ordered statistics of n iid uniformly distributed random variables in $(0,t)$. Let $\{Y_i, i \in [n]\}$ be iid non-negative random variables independent of $\tau_1, \tau_2, \ldots, \tau_n$. Then for $y \in (0,t)$, we have*

$$\Pr\left\{\sum_{i\in[k]} Y_i \leq \tau_k, k \in [n] \Big| \sum_{i\in[n]} Y_i = y\right\} = 1 - \frac{y}{t}. \tag{2}$$

Proof. We will prove this by induction on n. For base step of $n = 1$, inductive hypothesis is true since

$$\Pr(Y_1 < \tau_1 | Y_1 = y) = \Pr(y < \tau_1).$$

We assume the inductive hypothesis to be true for $n-1$. Defining $T_k = \sum_{i\in[k]} Y_i$, and using Lemma 1.9, we can write

$$\Pr\left\{T_k \leq \tau_k, k \in [n] \Big| T_n = y, T_{n-1} = s, \tau_n = u\right\} = \Pr\left\{T_k \leq \tau_k^*, k \in [n-1] \Big| T_{n-1} = s\right\}.$$

where τ_k^* are order statistics of $n-1$ iid uniform random variables in $[0,u)$. From inductive hypothesis, it follows that

$$\Pr\left\{T_k \leq \tau_k, k \in [n] \Big| T_n, T_{n-1}, \tau_n\right\} = \left(1 - \frac{T_{n-1}}{\tau_n}\right) 1_{\{T_n < \tau_n\}}.$$

Using Lemma 1.8, we can write

$$\Pr\left\{T_k \leq \tau_k, k \in [n] \middle| T_n, \tau_n\right\} = \mathbb{E}\left[\left(1 - \frac{T_{n-1}}{\tau_n}\right) 1_{\{T_n \leq \tau_n\}} \middle| T_n, \tau_n\right] = \left(1 - \frac{(n-1)T_n}{n\tau_n}\right) 1_{\{T_n \leq \tau_n\}}.$$

Since τ_n is maximum of n uniform random variables distributed on $[0, t)$, we can write

$$\Pr\{\tau_n \leq x\} = \left(\frac{x}{t}\right)^n, \text{ for } x \in [0, t).$$

Hence, by taking expectation with respect to τ_n, we can write

$$\begin{aligned}\Pr\left\{T_k \leq \tau_k, k \in [n] \middle| T_n\right\} &= \Pr\{T_n \leq \tau_n\} - \frac{n-1}{n} T_n \mathbb{E}\left[\frac{1}{\tau_n} 1_{\{T_n \leq \tau_n\}}\right] \\ &= 1 - \left(\frac{T_n}{t}\right)^n - (n-1)T_n \int_{T_n}^{t} \frac{x^{n-2}}{t^n} dx = 1 - \frac{T_n}{t}.\end{aligned}$$

□

Lemma 1.11. *Let $\tau_1, \tau_2, \ldots, \tau_{n-1}$ be the order statistics of $n-1$ iid random variables distributed uniformly in $[0, t)$. Let $\{Y_i, i \in [n]\}$ be iid nonnegative random variables that are also independent of $\tau_1, \tau_2, \ldots, \tau_{n-1}$. Then*

$$\Pr(\sum_{i \in [k]} Y_i < \tau_k, k \in [n-1] | \sum_{i \in [n]} Y_i = t) = \frac{1}{n}$$

Proof. Using definition of $T_k = \sum_{i \in [k]} Y_i$ and Lemma 1.10, we can write

$$\Pr\left\{T_k < \tau_k, k \in [n-1] \middle| T_n = t, T_{n-1}\right\} = \Pr\left\{T_k < \tau_k, k \in [n-1] \middle| T_{n-1}\right\} = \left(1 - \frac{T_{n-1}}{t}\right) \mathbf{1}_{\{T_{n-1} < t\}}.$$

Hence using Lemma 1.8, we have

$$\Pr\left\{T_k < \tau_k, k \in [n-1] \middle| T_n = t\right\} = \mathbb{E}\left[\left(1 - \frac{T_{n-1}}{t}\right) \mathbf{1}_{\{T_{n-1} < t\}} \middle| T_n = t\right] = 1 - \frac{(n-1)t}{nt} = \frac{1}{n}.$$

□

2 Compound Poisson Process

Definition 2.1. A stochastic Process $\{Z_t, t \geqslant 0\}$ is said to be a compound Poisson Process if it can be represented as $Z_t = \sum_{i=1}^{N_t} X_i$ for all $t \geq 0$ where $\{N_t, t \geq 0\}$ is a Poisson Process and $\{X_i, i \in \mathbb{N}\}$ are iid random variables independent of $\{N_t, t \geq 0\}$.

Alternately it can also be defined in the following way.

Definition 2.2. A compound Poisson Process is a point Process $\{Z_t, t \geq 0\}$ having the following properties.

1. For all $\omega \in \Omega, t \longmapsto Z_t(\omega)$ has finitely many jumps in finite intervals.
2. For all $t, s \geq 0; Z_{t+s} - Z_t$ is independent of $\{Z_u, u \leq t\}$.
3. For all $t, s \geq 0$, distribution of $Z_{t+s} - Z_t$ depends only on s and not on t.

Definition 2.3. A compound Poisson Process is stationary and independent increments point Process with jump points $S_n = \inf\{t > 0 | N(t) = n\}$, and associated jump sizes X_n independent of jump instants.

2.1 Examples

Example 2.4. Arrival of customers in a store is a Poison Process N_t. Let the amount spent by each customer be iid random variables independent of the arrival Process. Amount of money spent by first n customers is

$$Y_n = \sum_{i=1}^{n} X_i, i \in [n], \qquad\qquad Y_0 = 0.$$

Now define $Z_t = Y_{N_t}$ as the amount spent by the customers arriving in time t. Then $\{Z_t, t \geq 0\}$ is a compound Poisson Process.

Example 2.5. Let the time between successive failures of a machine be independent and exponentially distributed. The cost of repair is iid random at each failure. Then the total cost of repair in a certain time t is a compound Poisson Process.

Example 2.6. Let $X_i \in E$ where E is a countable set. Let N_t^e be the number of jumps of size e in time $[0, t)$.

Remark 2.7. Observe that $\{N_t^e, e \in E\}$ are independent Poisson with rate $\{\lambda_e : e \in E\}$ where $\lambda_e = \lambda \Pr\{X_i = e\}$.

We define $Z_t = \sum_{e \in E} e N_t^e$, to obtain

$$\mathbb{E}\left[e^{-\theta Z_t}\right] = \mathbb{E}\left[\prod_{e \in E} e^{-\theta e N_t^e}\right] = \prod_{e \in E} \mathbb{E}\left[e^{-\theta e N_t^e}\right] \tag{3}$$

However, we have

$$\mathbb{E}\left[e^{-\theta N_t}\right] = \sum_{n=0}^{\infty} e^{-\theta n} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{-\lambda t (1 - e^{-\theta})}.$$

Substituting this back in equation (3), we obtain

$$\begin{aligned}\mathbb{E}\left[e^{-\theta Z_t}\right] &= \prod_{e \in E} e^{-\lambda_e t (1 - e^{-\theta e})} \\ &= \exp\left[-t \sum_{e \in E} \lambda (1 - e^{-\theta e}) \Pr\{X_i = e\}\right]\end{aligned}$$

Example 2.8. If X_i iid with mean μ, then we can find mean off Z_t as

$$\mathbb{E}\left[Z_t\right] = \mathbb{E}\left[\mathbb{E}[Z_t | N_t]\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_t} X_i | N_t\right]\right] = \mu \mathbb{E} N_t = \lambda \mu t.$$

Example 2.9. If X_i are iid with a distribution function φ, then we can write moment generating function of Z_t in terms of moment generating function $f(\theta) = E[e^{-\theta X_1}]$ of X_1 as

$$\begin{aligned}\mathbb{E}\left[e^{-\theta Z_t}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-\theta \sum_{i=1}^{n-1} X_i}\right] | N_t\right] = \sum_{n=0}^{\infty} e^{\lambda t} \frac{(-\lambda t)^n}{n!} f(\theta)^n = e^{-\lambda t (1 - f(\theta))} \\ &= \exp\left[-t \int_0^{\infty} (1 - e^{-\theta u}) \lambda d\varphi(u)\right].\end{aligned}$$

2.1.1 A nice counterexample

A Poisson Process is not uniquely determined by it's distribution. Let $X_t = Y_t + f(Z+t)$, where Y_t is a Poisson Process and

$$f(t) = t1_{\{t\in\mathbb{Q}\}}.$$

Let Z be a continuous random variable. Then we can show that $\Pr\{X_t \neq Y_t) = 0$. This is true since

$$\begin{aligned}\Pr\{X_t \neq Y_t\} &= \Pr\{\omega \in \Omega : \quad t + Z(\omega) \in \mathbb{Q}\}\\ &= \Pr\{\omega \in \Omega : Z(\omega) \in \mathbb{Q} - t\} = 0.\end{aligned}$$

The last part follows since $\mathbb{Q} - t$ is a countable set of individual events with Probability zero. We can also show that $X(t)$ and $Y(t)$ have same fdds.

$$\Pr\{X_{t_1} = Y_{t_1}, X_{t_2} = Y_{t_2}\} = \sum_{n_1,n_2} \Pr\{X_{t_1} = n_1, X_{t_2} = n_2, Y_{t_1} = n_1, Y_{t_2} = n_2\} = 1.$$

$\{X_t(\omega)\}$ can take non-integer values and is not non-decreasing. Two Process can have same distribution but sample path behavior can be quite different.

3 Non-Homogeneous Poisson Process

From the characterization of Poisson Process just stated, we can generalize to non-homogeneous Poisson Process. In this case, the rate of Poisson Process λ is time varying. It is not clear from the first two characterizations, how to generalize the definition of Poisson Process to the non-homogeneous case. We used third characterization of Poisson Process for this generalization.

Definition 3.1 (Non-Homogeneous Poisson Process). A point Process $\{N(t),\ t \geqslant 0\}$ is said to be **non-homogeneous Poisson Process** with instantaneous rate $m(t)$ if it has stationary independent increments, and

$$\begin{aligned}\Pr\{N(t) = 0\} &= 1 - m(t) + o(t).\\ \Pr\{N(t+\delta) - N(t) = 0\} &= 1 - m(t)\delta + o(\delta).\\ \Pr\{N(t+\delta) - N(t) = 1\} &= m(t)\delta + o(\delta).\\ \Pr\{N(t+\delta) - N(t) > 1\} &= o(\delta).\end{aligned}$$

Proposition 3.2 (Non-Homogeneous Distribution). *Distribution of non-homogeneous Poisson Process $N(t)$ with instantaneous rate $m(t)$ is given by*

$$\Pr\{N(t) = n\} = \frac{(\bar{m}(t))^n}{n!} e^{-\bar{m}(t)},$$

where $\bar{m}(t)$ is the cumulative rate till time t, i.e. $\bar{m}(t) = \int_0^t m(s)ds$.

Proof. Let's denote $f(t) = \Pr\{N(t) = 0\}$. Further, from independent increment property of $N(t)$, we notice that $\{N(t+\delta) = 0\}$ is intersection of two independent events given below,

$$\{N(t+\delta) = 0\} \iff \{N(t) = 0\} \cap \{N(t+\delta) - N(t) = 0\}.$$

From Definition 3.1, it follows that

$$f(t+\delta) = f(t)[1 - m(t)\delta + o(\delta)].$$

Re-arranging the terms in the above align, dividing by δ, and taking limit as $\delta \downarrow 0$, we get

$$f'(t) = -m(t)f(t).$$

Since $f(0) = 1$, it can be verified that $f(t) = \exp(-\bar{m}(t))$ is solution for $f(t)$. We have shown $\Pr\{N(t) = 0\} = \exp(-\bar{m}(t))$. By induction, we can show the result for any n. $\square$