Lecture 05: Renewal Theory

1 Renewal Theory

One of the characterization for the Poisson process is of it being a counting process with iid exponential inter-arrival times. Now we shall relax the "exponential" part.

Definition 1.1. A counting process $\{N(t), t \geq 0\}$ with iid general inter-arrival times is called a **renewal process**.

As a result, we no longer have the nice properties such as Independent and stationary increments that Poisson processes had. However, we can still get some great results which also apply to Poisson Processes.

Definition 1.2 (Inter-arrival Times). Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of non-negative iid random variables with a common distribution F, with

1. finite mean μ,
2. $F(0) < 1$.

Second condition implies non-degenerate renewal process, if $F(0)$ is equal to 1 then it is a trivial process. We interpret X_n as the time between $(n-1)^{\text{st}}$ and the n^{th} renewal event.

Definition 1.3 (Renewal Instants). Let S_n denote the time of n^{th} renewal, and assume $S_0 = 0$. Then, we have

$$S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}.$$

Definition 1.4 (Renewal process). Let $\{N(t), t \geq 0\}$ be the counting process that counts number of events by time t. Then,

$$N(t) = \sup\{n \in \mathbb{N}_0 : S_n \leq t\} = \sum_{n \in \mathbb{N}} 1_{\{S_n \leq t\}}.$$

This counting process $\{N(t), t \geq 0\}$ is called a renewal process.

Lemma 1.5 (Inverse Relationship). *There is an inverse relationship between time of n^{th} event S_n, and the counting process $N(t)$. That is*

$$\{S_n \leq t\} \iff \{N(t) \geq n\}. \tag{1}$$

Lemma 1.6 (Finiteness of $N(t)$). *We are interested in knowing how many renewals occur per unit time. From SLLN, we have*

$$\frac{S_n}{n} \to \mu \quad a.s.$$

Since $\mu > 0$, we must have S_n growing arbitrarily large as n increases. Thus, S_n can be finite for at most finitely many n. Therefore, $N(t)$ must be finite, and

$$N(t) = \max\{n \in \mathbb{N}_0 : S_n \leq t\}.$$

1.1 Distribution of N(t)

We need to know the distribution of $N(t)$.

Lemma 1.7. *Counting process $N(t)$ assumes non-negative integer values with distribution*

$$\Pr\{N(t) = n\} = \Pr\{S_n \leq t\} - \Pr\{S_{n+1} \leq t\} = F_n(t) - F_{n+1}(t).$$

Proof. It follows from (**??**). □

Definition 1.8. Let F_n be the distribution of renewal instant S_n i.e. $\Pr\{S_n \leq t\} = F_n(t)$.

Lemma 1.9. *Distribution F_n of renewal instant S_n is given inductively by*

$$F_1 = F, \qquad F_n = F_{n-1} * F,$$

where $*$ denotes convolution.

Proof. It follows from induction over sum of iid random variables. □

Definition 1.10. We define $m(t) = \mathbb{E}[N(t)]$ to be the **renewal function**.

Proposition 1.11. *Renewal function can be expressed in terms of distribution of renewal instants as*

$$m(t) = \sum_{n \in \mathbb{N}} F_n(t).$$

Proof.

$$\begin{aligned} m(t) &= \mathbb{E}[N(t)] \\ &= \sum_{n \in \mathbb{N}} \Pr\{N(t) \geq n\} \\ &= \sum_{n \in \mathbb{N}} \Pr\{S_n \leq t\} = \sum_{n \in \mathbb{N}} F_n(t). \end{aligned}$$

where the second equality follows from the fact that the expectation of a random variable being represented in terms of the ccdf of the corresponding random variable, the third equality follows from the inverse relationship as seen in (**??**).
Alternatively,

$$\begin{aligned} m(t) &= \mathbb{E}[N(t)]. \\ &= \mathbb{E}\left[\sum_{n \in \mathbb{N}} \mathbb{I}_{\{S_n \leq t\}}\right] \\ &= \sum_{n \in \mathbb{N}} \mathbb{E}\left[\mathbb{I}_{\{S_n \leq t\}}\right] \\ &= \sum_{n \in \mathbb{N}} \Pr\{S_n \leq t\} = \sum_{n \in \mathbb{N}} F_n(t). \end{aligned}$$

where the third equality follows from the Monotone Convergence Theorem. □

Proposition 1.12. *Renewal function is bounded for all finite times.*

Proof. Since we assumed that $\Pr\{X_n = 0\} < 1$, it follow from continuity of probabilities that there exists $\alpha > 0$, such that $\Pr\{X_n \geq \alpha\} > 0$. Define

$$\bar{X}_n = \alpha 1_{\{X_n \geq \alpha\}}.$$

Let $\bar{N}(t)$ denote the renewal process with inter-arrival times $\bar{X}_n$. Note that since X_i's are iid, so are $\bar{X}_i$ (which will be evident from the proof of the distribution function of the number of arrivals till time t). In fact, the arrivals now happen at multiples of α. And yes, they stack. Moreover, $X_n \geq \bar{X}_n$.

$$\begin{aligned} Pr\{Number\ of\ arrivals\ at\ time\ 0\ \ = n\} &= Pr\{\bar{X_1} = \bar{X_2} = \ldots = \bar{X_n} = 0, \bar{X_{n+1}} = \alpha\} \\ &= Pr\{X_1 < \alpha, X_2 < \alpha, \ldots, X_n < \alpha, X_{n+1} \geq \alpha\} \\ &= \prod_{i=1}^{n} Pr\{X_i < \alpha\}.Pr\{X_{n+1} \geq \alpha\} \\ &= (1 - Pr\{X_1 \geq \alpha\})^n .Pr\{X_1 \geq \alpha\}. \end{aligned}$$

where the third equality follows from the fact that $X_i, i \in \mathbb{N}$ are mutually independent, fourth equality follows from the fact that $X_i, i \in \mathbb{N}$ are identical.
The number of arrivals till time t therefore is Geometric with mean $\frac{1}{P[X_n \geq \alpha]}$. Thus

$$\mathbb{E}[\bar{N}(t)] = \frac{\lfloor \frac{t}{\alpha} \rfloor + 1}{P[X_n \geq \alpha]} \leq \frac{\frac{t}{\alpha} + 1}{P[X_n \geq \alpha]} < \infty.$$

Since $\mathbb{E}[N(t)] \leq \mathbb{E}[\bar{N}(t)]$ which follows from $N(t) \leq \bar{N}(t)$, we are done. □

2 Limit Theorems

Lemma 2.1. *Let* $N(\infty) := \lim_{t\to\infty} N(t)$. *Then, it is easy to see that* $\Pr\{N(\infty) = \infty\} = 1$.

Proof. It suffices to show $\Pr\{N(\infty) < \infty\} = 0$. We have

$$\begin{aligned} \Pr\{N(\infty) < \infty\} &= \Pr\{\bigcup_{n\in\mathbb{N}} \{N(\infty) < n\}\} \\ &= \Pr\{\bigcup_{n\in\mathbb{N}} \{S_n = \infty\}\} = \Pr\{\bigcup_{n\in\mathbb{N}} \{X_n = \infty\}\} \\ &\leq \sum_{n\in\mathbb{N}} \Pr\{X_n = \infty\} = 0. \end{aligned}$$

The last step follows from the fact that $\mathbb{E}[X_n] < \infty$. □

Theorem 2.2 (Basic Renewal Theorem).

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \textit{almost surely.}$$

[Notice that $N(t)$ increases to infinity with time. We are interested in rate of increase of $N(t)$ with t. Note that $S_{N(t)}$ represents the time of last renewal before t, and $S_{N(t)+1}$ represents the time of first renewal after time t.]

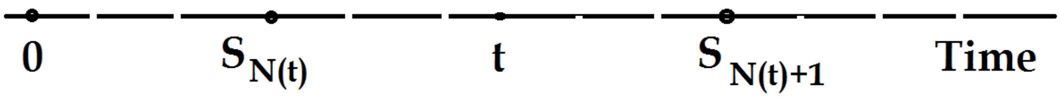

Figure 1: Time-line visualization

Proof. Consider $S_{N(t)}$. By definition, we have

$$S_{N(t)} \leq t < S_{N(t)+1}$$

Dividing by $N(t)$, we get

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

By Strong Law of Large Numbers (SLLN) and the previous result, we have

$$\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \mu \quad \text{a.s.}$$

Also

$$\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)} = \lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)+1} . \frac{N(t)+1}{N(t)}$$

Hence by Squeeze Theorem, the result follows. □

2.0.1 Example

Suppose, you are in a casino with infinitely many games. Every game has a probability of win X, iid uniformly distributed between $(0, 1)$. One can continue to play a game or switch to another one. We are interested in a strategy that maximizes the long-run proportion of wins.

Let $N(n)$ denote the number of losses in n plays. Then fraction of wins $P_W(n)$ is given by

$$P_W(n) = \frac{n - N(n)}{n}.$$

We pick a strategy where any game is selected to play, and continue to be played till the first loss. Note that, time till first loss is geometrically distributed with mean $\frac{1}{1-X}$. We shall show that this fraction approaches unity as $n \to \infty$. By the previous proposition, we have:

$$\begin{aligned}\lim_{n\to\infty} \frac{N(n)}{n} &= \frac{1}{\mathbb{E}[\text{Time till first loss}]} \\ &= \frac{1}{\mathbb{E}\left[\frac{1}{1-X}\right]} = \frac{1}{\infty} = 0\end{aligned}$$

Hence Renewal theorems can be used to compute these long term averages. We'll have many such theorems in the following sections.

2.1 Wald's Lemma

Before we get into Wald's Lemma, let us first define what a stopping time is.

Definition 2.3 (Stopping Time). Let $\{X_n : n \in \mathbb{N}\}$ be independent random variables. Then T, an integer random variable, is called a stopping time with respect to this sequence if $\{N = n\}$ depends only on $\{X_1, \cdots, X_n\}$ and is independent of $\{X_{n+1}, X_{n+2}, \cdots\}$.

Intuitively, if we observe the X_n's in sequential order and N denotes the number observed before stopping then. Then, we have stopped after observing, $\{X_1, \ldots, X_N\}$, and before observing $\{X_{N+1}, X_{N+2}, \ldots\}$. The intuition behind a stopping time is that it's value is determined by past and present events but NOT by future events.

Example 2.4. For instance, while traveling on the bus, the random variable measuring "Time until bus crosses Majestic and after that one stop" is a stopping time as it's value is determined by events before it happens. On the other hand "Time until bus stops before Majestic is reached" would not be a stopping time in the same context. This is because we have to cross this time, reach Majestic and then realise we have crossed that point.

Example 2.5. Consider $X_n \in \{0, 1\}$ iid Bernoulli(1/2). Then $N = min\{n \in \mathbb{N} : \quad \sum_{i=1}^{n} X_i = 1\}$ is a stopping time. For instance, $Pr\{N = 2\} = Pr\{X_1 = 0, X_2 = 1\}$ and hence N is a stopping time by definition.

Example 2.6 (Random Walk Stopping Time). Consider X_n iid bivariate random variables with

$$\Pr\{X_n = 1\} = \Pr\{X_n = -1\} = \frac{1}{2}.$$

Then $N = min\{n \in \mathbb{N} : \quad \sum_{i=1}^{n} X_i = 1\}$ is a stopping time.

Properties of stopping time: Let N_1, N_2 be two stopping times with respect to $\{X_i : i \in \mathbb{N}\}$ then,

- $N_1 + N_2$ is a stopping time.
- $\min\{N_1, N_2\}$ is a stopping time.

Proof.

- $$\begin{aligned}\{N_1 + N_2 \le n\} &= \bigcup_{i=0}^{n} \{N_1 + N_2 = i\} \\ &= \bigcup_{i=0}^{n} \bigcup_{k=0}^{i} \{N_1 = k\} \cap \{N_2 = i - k\} \\ &\perp\!\!\!\perp \{X_{n+1}, X_{n+2}, \ldots\}.\end{aligned}$$

  Hence, $N_1 + N_2$ is a stopping time.

- $$\begin{aligned}\{\min\{N_1, N_2\} > n\} &= \{N_1 > n\} \cap \{N_2 > n\}. \\ \textit{From De Morgan's Law we get } \{\min\{N_1, N_2\} \le n\} &= \{N_1 \le n\} \cap \{N_2 \le n\} \\ &\perp\!\!\!\perp \{X_{n+1}, X_{n+2}, \ldots\}.\end{aligned}$$

  Hence, $\min\{N_1, N_2\}$ is a stopping time.

□

Lemma 2.7 (Wald's Lemma). *Let* $\{X_i : \quad i \in \mathbb{N}\}$ *be iid random variables with finite mean* $\mathbb{E}[X_1]$ *and let* N *be a stopping time with respect to this set of variables, such that* $\mathbb{E}[N] < \infty$. *Then,*

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}[X_1]\mathbb{E}[N]$$

Proof.

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}\left[\sum_{n\in\mathbb{N}} X_n 1_{\{N\geq n\}}\right] \tag{2}$$

$$= \sum_{n\in\mathbb{N}} \mathbb{E}\left[X_n 1_{\{N\geq n\}}\right]. \tag{3}$$

I'd like to point out here that in step (**??**), you cannot always exchange infinite sums and expectations. But here you can do so, because of the application of Monotone Convergence Theorem. Refer Ross/Wolff for more information. Regardless, to proceed, we need to show that $N \geq n$ is independent of $X_k, \quad k \geq n$. To this end, observe that

$$\{N \geq k\} = \{N < k\}^c = \{N \leq k-1\}^c = \left(\bigcup_{i=1}^{k-1}\{N = i\}\right)^c.$$

Since, N is a stopping time and by definition $\{N = i\}$ depends only on $\{X_1, \ldots, X_i\}$. Therefore, $\{N \geq k\}$ depends only on $\{X_1, \ldots, X_{k-1}\}$, and is independent of the future and present samples. Therefore, we can write

$$\begin{aligned}\sum_{n\in\mathbb{N}} \mathbb{E}\left[X_n 1_{\{N\geq n\}}\right] &= \sum_{n\in\mathbb{N}} \mathbb{E}\left[X_n\right] \mathbb{E}\left[1_{\{N\geq n\}}\right] \\ &= \mathbb{E}\left[X_1\right] \sum_{n\in\mathbb{N}} \Pr\{N \geq n\} \\ &= \mathbb{E}[X_1]\mathbb{E}[N].\end{aligned}$$

where the third equality follows from the fact that the expectation of a random variable being represented in terms of the ccdf of the corresponding random variable. □

Proposition 2.8 (Wald's Lemma for Renewal Process). *Let* $\{X_n, n \in \mathbb{N}\}$ *be iid inter-arrival times of a renewal process* $N(t)$ *with* $\mathbb{E}[X_1] < \infty$, *and let* $m(t) = \mathbb{E}[N(t)]$ *be its renewal function. Then,* $N(t) + 1$ *is a stopping time and*

$$\mathbb{E}\left[\sum_{i=1}^{N(t)+1} X_i\right] = \mathbb{E}[X_1][1 + m(t)].$$

Proof. It is easy to see that $\{N(t) + 1 = n\}$ depends solely on $\{X_1, \ldots, X_n\}$ from the discussion below.

$$\{N(t) + 1 = n\} \iff \{S_{n-1} \leq t < S_n\} \iff \left\{\sum_{i=1}^{n-1} X_i \leq t < \sum_{i=1}^{n-1} X_i + X_n\right\}.$$

Thus $N(t) + 1$ is a stopping time, and the result follows from Wald's Lemma. □

2.2 Elementary Renewal Theorem

Basic renewal theorem implies $N(t)/t$ converges to $1/\mu$ almost surely. Now, we are interested in convergence of $\mathbb{E}[N(t)]/t$. Note that this is not obvious, since almost sure convergence doesn't imply convergence in mean. Consider the following example.

Example 2.9.

$$Y_n = \begin{cases} n, & \text{w.p. } 1/n, \\ 0, & \text{w.p. } 1 - 1/n. \end{cases}$$

Then, $\Pr\{Y_n = 0\} = 1 - 1/n$. That is $Y_n \to 0$ a.s. However, $\mathbb{E}[Y_n] = 1$ for all $n \in \mathbb{N}$. So $\mathbb{E}[Y_n] \to 1$.

Even though, basic renewal theorem does **NOT** imply it, we still have $\mathbb{E}[N(t)]/t$ converging to $1/\mu$.

Theorem 2.10 (Elementary Renewal Theorem). *Let $m(t)$ denote mean $\mathbb{E}[N(t)]$ of renewal process $N(t)$, then under the hypotheses of basic renewal theorem, we have*

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

Proof. Take $\mu < \infty$. We know that $S_{N(t)+1} > t$. Therefore, taking expectations on both sides and using Proposition **??**, we have

$$\mu(m(t) + 1) > t.$$

Dividing both sides by μt and taking lim inf on both sides, we get

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}. \tag{4}$$

We employ a truncated random variable argument to show the reverse inequality. We define truncated inter-arrival times $\{\bar{X}_n\}$ as

$$\bar{X}_n = X_n 1_{\{X_n \leq M\}} + M 1_{\{X_n > M\}}.$$

We will call $\mathbb{E}[\bar{X}_n] = \mu_M$. Further, we can define arrival instants $\{\bar{S}_n\}$ and renewal process $\bar{N}(t)$ for this set of truncated inter-arrival times $\{\bar{X}_n\}$ as

$$\bar{S}_n = \sum_{k=1}^{n} \bar{X}_k, \qquad \bar{N}(t) = \sup\{n \in \mathbb{N}_0 : \bar{S}_n \leq t\}.$$

Note that since $S_n \geq \bar{S}_n$, number of arrivals would be higher for renewal process with truncated random variables, i.e.

$$N(t) \leq \bar{N}(t). \tag{5}$$

Further, due to truncation of inter-arrival time, next renewal happens with-in M units of time, i.e.

$$\bar{S}_{N(t)+1} \leq t + M.$$

Taking expectations on both sides in the above align, using Proposition **??**, dividing both sides by $t\mu_M$ and taking lim sup on both sides, we obtain

$$\limsup_{t\to\infty} \frac{\bar{m(t)}}{t} \leq \frac{1}{\mu_M}.$$

Taking expectations on both sides of (**??**) and letting M go arbitrary large on RHS, we get

$$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}. \tag{6}$$

Result for finite μ follows from (**??**) and (**??**). When μ grows arbitrary large, results follow from (**??**), where RHS is zero. □

2.3 Central Limit for Renewal Processes

Theorem 2.11. *Let X_n be iid random variables with $\mu = \mathbb{E}[X_n] < \infty$ and $\sigma^2 = Var(X_n) < \infty$. Then*

$$\frac{N(t) - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} \to^d N(0,1)$$

Proof. Take $u = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$. We shall treat u as an integer and proceed, the proof for general u is an exercise. Recall that $\{N(t) < u\} \iff \{S_u > t\}$. By equating probability measures on both sides, we get

$$\Pr\{N(t) < u\} = \Pr\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > \frac{t - u\mu}{\sigma\sqrt{u}}\right\} = \Pr\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > -y\left(1 + \frac{y\sigma}{\sqrt{tu}}\right)^2\right\}.$$

By central limit theorem, $\frac{S_u - u\mu}{\sigma\sqrt{u}}$ converges to a normal random variable with zero mean and unit variance as t grows. Also, note that

$$\lim_{t\to\infty} -y\left(1 + \frac{y\sigma}{\sqrt{tu}}\right)^2 = -y.$$

These results combine with the symmetry of normal random variable to give us the result. □