Lecture 06: Key Renewal Theorem and Applications

1 Key Renewal Theorem and Applications

Definition 1.1 (Lattice Random Variable). A non-negative random variable X is said to be **lattice** if there exists $d \geq 0$ such that

$$\sum_{n \in \mathbb{N}} \Pr\{X = nd\} = 1.$$

For a lattice X, its period is defined as

$$d = \sup\{d \in \mathbb{R}^+ : \sum_{n \in \mathbb{N}} \Pr\{X = nd\} = 1\}.$$

If X is a lattice random variable, distribution function F is also called lattice.

Theorem 1.2 (Blackwell's Theorem). *Let $N(t)$ be a renewal process with mean $m(t)$, and inter-arrival times with distribution F and mean μ. If F is not lattice, then for all $a \geq 0$*

$$\lim_{t \to \infty} m(t+a) - m(t) = \frac{a}{\mu}.$$

If F is lattice with period d, then

$$\lim_{n \to \infty} \mathbb{E}[\textit{number of renewals at } nd] = \frac{d}{\mu}.$$

Proof. We will not prove that

$$g(a) = \lim_{t \to \infty} [m(t+a) - m(t)] \tag{1}$$

exists for non-lattice F. However, we show that if this limit does exist, it is equal to a/μ as a consequence of elementary renewal theorem. To this end, note that

$$m(t+a+b) - m(t) = m(t+a+b) - m(t+a) + m(t+a) - m(t).$$

Taking limits on both sides of the above equation, we conclude that $g(a+b) = g(a) + g(b)$. The only increasing solution of such a g is

$$g(a) = ca, \forall a > 0,$$

for some positive constant c. To show $c = \frac{1}{\mu}$, define a sequence $\{x_n, n \in \mathbb{N}\}$ in terms of $m(t)$ as

$$x_n = m(n) - m(n-1),\ n \in \mathbb{N}.$$

Note that $\sum_{i=1}^n x_i = m(n)$ and $\lim_{n\in\mathbb{N}} x_n = g(1) = c$, hence we have

$$\lim_{n\in\mathbb{N}} \frac{\sum_{i=1}^n x_i}{n} = \lim_{n\in\mathbb{N}} \frac{m(n)}{n} \overset{(a)}{=} c,$$

where (a) follows from the fact that if a sequence $\{x_i\}$ converges to c, then the running average sequence $a_n = \frac{1}{n}\sum_{i=1}^n x_i$ also converges to c, as $n \to \infty$.

Therefore, we can conclude $c = 1/\mu$ by elementary renewal theorem.

When F is lattice with period d, the limit in (**??**) doesn't exist. (See Example **??**). However, the theorem is true for lattice trivially by elementary renewal theorem. □

Example 1.3. For a trivial lattice example where the $\lim_{t\to\infty} m(t+a) - m(t)$ does not exist, consider a renewal process with $\Pr\{X_n = 1\} = 1$, that is, there is a renewal at every positive integer time instant with probability 1. Then F is lattice with $d = 1$. Now, for $a = 0.5$, and $t_n = n + (-1)^n 0.5$, we see that $\lim_{t_n\to\infty} m(t_n + a) - m(t_n)$ does not exist, and hence $\lim_{t\to\infty} m(t+a) - m(t)$ does not exist.

1.1 Directly Riemann Integrable

Definition 1.4. A function $h : [0,\infty] \to \mathbb{R}$ is **directly Riemann integrable** if the partial sums obtained by summing the infimum and supremum of h, taken over intervals obtained by partitioning the positive axis, are finite and both converge to the same limit, for all finite positive interval lengths. That is,

$$\lim_{\delta\to 0} \delta \sum_{n\in\mathbb{N}} \sup_{u\in[(n-1)\delta, n\delta]} h(u) = \lim_{\delta\to 0} \delta \sum_{n\in\mathbb{N}} \inf_{u\in[(n-1)\delta, n\delta]} h(u)$$

If both limits exist and are equal, then the integral value is equal to the limit.

Compare this definition with the definition of Riemann integrals. A function $g : [0, M] \to \mathbb{R}$ for $0 < M < \infty$ is Riemann integrable if

$$\lim_{\delta\to 0} \delta \sum_{k=0}^{M/\delta} \sup_{u\in[(n-1)\delta, n\delta]} g(u) = \lim_{\delta\to 0} \delta \sum_{k=0}^{M/\delta} \inf_{u\in[(n-1)\delta, n\delta]} g(u)$$

and in that case, limit is the value of the integral. For h defined on $[0,\infty]$, $\int_0^\infty h(u)du = \lim_{M\to\infty} \int_0^M h(u)du$, if the limit exists. For many functions, this limit may not exist.

Remark 1.5. A directly Riemann integral function over $[0,\infty)$ is also Riemann integral, but the converse need not be true. For instance $h(t) = \sum_{n\in\mathbb{N}} 1_{\left[n-\frac{1}{(2n^2)}, n+\frac{1}{(2n^2)}\right]}(t)$ is Riemann integral, but $\delta \sum_{n\in\mathbb{N}} \sup_{u\in[(n-1)\delta, n\delta]} h(u)$ is always infinite for every $\delta > 0$.

Proposition 1.6. *Following are sufficient conditions for a function h to be directly Riemann integrable.*

1. *If h is bounded and continuous and h is non increasing.*
2. *If h is bounded above by a directly Riemann integrable function.*
3. *If h is non-negative, non-increasing, and with bounded integral.*

Proposition 1.7 (Tail Property). *If h is non-negative, directly Riemann integrable, and has bounded integral value, then*

$$\lim_{t\to\infty} h(t) = 0.$$

Theorem 1.8 (Key Renewal Theorem). *Let $N(t)$ be a renewal process having mean $m(t)$, and iid inter-arrival times with mean μ and distribution function F. If F is non-lattice, and if a function $h(t)$ is directly Riemann integrable, then*

$$\lim_{t\to\infty}\int_0^\infty h(t-x)dm(x) = \frac{1}{\mu}\int_0^\infty h(t)dt, \tag{2}$$

where

$$m(t) = \sum_{n\in\mathbb{N}} F_n(t), \qquad \mu = \int_0^\infty \bar{F}(t).$$

Proposition 1.9 (Equivalence). *Blackwell's theorem and key renewal theorem are equivalent.*

Proof. Let's assume key renewal theorem is true. We select h as a simple function with value unity on interval $[0,a]$ and zero elsewhere. That is,

$$h(x) = 1_{\{x\in[0,a]\}}.$$

It is easy to see that this function is directly Riemann integrable.

To see how we can prove the key renewal theorem from Blackwell's theorem, observe from Blackwell's theorem that,

$$\lim_{t\to\infty}\frac{dm(t)}{dt} \overset{(a)}{=} \lim_{a\to 0}\lim_{t\to\infty}\frac{m(t+a)-m(t)}{a} = \frac{1}{\mu}.$$

where in (a) we can exchange the order of limits under certain regularity conditions. We defer the formal proof for a later stage. □

Remark 1.10. Key renewal theorem is very useful in computing the limiting value of some function $g(t)$, probability or expectation of an event at an arbitrary time t, for a renewal process. This value is computed by conditioning on the time of last renewal prior to time t.

Theorem 1.11 (Key Lemma). *Let $N(t)$ be a renewal process, with mean $m(t)$, iid inter-renewal times $\{X_n\}$ with distribution function F, and n^{th} renewal instant S_n. Then,*

$$\Pr\{S_{N(t)} \le s\} = \bar{F}(t) + \int_0^s \bar{F}(t-y)dm(y), \qquad t \ge s \ge 0.$$

Proof. We can see that event of time of last renewal prior to t being smaller than another time s can be partitioned into disjoint events corresponding to number of renewals till time t. Each of these disjoint events is equivalent to occurrence of n^{th} renewal before time s and $(n+1)^{\text{st}}$ renewal past time t. That is,

$$\{S_{N(t)} \le s\} = \bigcup_{n\in\mathbb{N}_0}\{S_{N(t)} \le s, N(t) = n\} = \bigcup_{n\in\mathbb{N}_0}\{S_n \le s, S_{n+1} > t\}.$$

Recognizing that $S_0 = 0$, $S_1 = X_1$, and that $S_{n+1} = S_n + X_{n+1}$, we can write

$$\Pr\{S_{N(t)} \le s\} = \Pr\{X_1 > t\} + \sum_{n\in\mathbb{N}}\Pr\{X_{n+1} + S_n > t, S_n \le s\}.$$

We recall F_n, n-fold convolution of F, is the distribution function of S_n. Conditioning on $\{S_n = y\}$, we can write

$$\begin{aligned}\Pr\{S_{N(t)} \le s\} &= \bar{F}(t) + \sum_{n\in\mathbb{N}} \int_{y=0}^{s} \Pr\{X_{n+1} > t - S_n, S_n \le s | S_n = y\} dF_n(y),\\ &= \bar{F}(t) + \sum_{n\in\mathbb{N}} \int_{y=0}^{s} \bar{F}(t-y) dF_n(y).\end{aligned}$$

Using monotone convergence theorem to interchange integral and summation, and noticing that $m(y) = \sum_{n\in\mathbb{N}} F_n(y)$, the result follows. □

Remark 1.12. Key lemma tells us that distribution of $S_{N(t)}$ has probability mass at 0 and density between $(0, t]$, that is,

$$\Pr\{S_{N(t)} = 0\} = \bar{F}(t), \qquad dF_{S_{N(t)}}(y) = \bar{F}(t-y)dm(y) \quad 0 < y \le t.$$

Remark 1.13. Density of $S_{N(t)}$ has interpretation of renewal taking place in the infinitesimal neighborhood of y, and next inter-arrival after time $t - y$. To see this, we notice

$$dm(y) = \sum_{n\in\mathbb{N}} dF_n(y) = \sum_{n\in\mathbb{N}} \Pr\{n^{\text{th}}\text{renewal occurs in}(y, y + dy)\}.$$

Combining interpretation of density of inter-arrival time $dF(t)$, we get

$$dF_{S_{N(t)}}(y) = \Pr\{\text{renewal occurs in } (y, y + dy) \text{ and next arrival after } t - y\}.$$

1.2 Alternating Renewal Processes

Alternating renewal processes form an important class of renewal processes, and model many interesting applications. We find one natural application of key renewal theorem in this section.

Definition 1.14 (Alternating Renewal Process). Let $\{(Z_n, Y_n),\ n \in \mathbb{N}\}$ be an iid random process, where Y_n and Z_n are not necessarily independent. A renewal process where each inter-arrival time X_n consist of ON time Z_n followed by OFF time Y_n, is called an **alternating renewal process**. We denote the distributions for ON, OFF, and renewal periods by H, G, and F, respectively. Let

$$P(t) = \Pr\{\text{ON at time } t\}.$$

Remark 1.15. To see that the alternating renewal process is indeed a renewal process, it needs to be established that $\{X_n : n \in \mathbb{N}\}$ is an iid sequence. But this trivially follows from the fact that $\{f(Y_n, Z_n) : n \in \mathbb{N}\}$ is an iid sequence whenever $\{(Z_n, Y_n),\ n \in \mathbb{N}\}$ is an iid sequence. Let $f(a, b) = a + b$ to see that $\{X_n : n \in \mathbb{N}\}$ is an iid sequence.

Theorem 1.16 (ON Probability). *If $\mathbb{E}[Z_n + Y_n] < \infty$ and F is non-lattice, then*

$$P(t) = \bar{H}(t) + \int_0^t \bar{H}(t-y)dm(y).$$

Proof. To find time dependent probability $P(t)$, we can partition the event of system being ON at time t on value of last renewal time $S_{N(t)}$. That is, we can write

$$\{\text{ON at time } t\} = \bigcup_{y\in[0,t)} \{\text{ON at time } t, S_{N(t)} = y\}.$$

Since any ON time is possibly only dependent on the corresponding OFF time and no past renewal times, conditioned on $\{S_{N(t)} = y\}$, the system stays ON at time t iff ON time is longer than $t - y$ conditioned on renewal time being larger than $t - y$. That is,

$$\{\text{ON at time } t | S_{N(t)} = y\} = \{Z_1 > t - y | Z_1 + Y_1 > t - y\}.$$

Since, for $y > 0$, we have $\Pr\{Z_1 > t - y | Z_1 + Y_1 > t - y\} = \frac{\bar{H}(t-y)}{\bar{F}(t-y)}$, it follows that

$$P(t) = \bar{H}(t) + \int_0^t \bar{H}(t-y)\bar{F}(t-y) dF_{S_{N(t)}}(y)$$

In view of the density of $S_{N(t)}$ from Remark **??**, the result follows. □

Corollary 1.17 (Limiting ON Probability). *If* $\mathbb{E}[Z_n + Y_n] < \infty$ *and* F *is non-lattice, then*

$$\lim_{t\to\infty} P(t) = \frac{\mathbb{E}[Z_n]}{\mathbb{E}[Y_n] + \mathbb{E}[Z_n]}.$$

Proof. Since H is the distribution function of the non-negative random variable Z_n, it follows that

$$\lim_{t\to\infty} \bar{H}(t) = 0, \text{ and } \int_0^\infty \bar{H}(t)dt = E[Z_n].$$

Applying key renewal theorem to Theorem **??**, we get the result. □

Many processes of practical interest can be modeled by an alternate renewal process.

Example 1.18 (Age and Excess Time). Consider a renewal process and let $A(t)$ be the time from t since the last renewal and $Y(t)$ be the time from t till the next renewal. That is,

$$Y(t) = S_{N(t)+1} - t,$$
$$A(t) = t - S_{N(t)}.$$

Suppose we need to find $\lim_{t\to\infty} \Pr\{A(t) \leq x\}$ for some fixed x. Now, observe that $\Pr\{A(t) \leq x\} = \mathbb{E}[1_{\{A(t)\leq x\}}]$ which is the mean time when the "age at t" is less than x which is equal to $\mathbb{E}[\min\{x, X\}]$. Hence, we get

$$\lim_{t\to\infty} \Pr\{A(t) \leq x\} = \frac{\mathbb{E}\min\{x, X\}}{\mathbb{E}X} = \frac{\int_0^x \bar{F}(t)dt}{\mu}.$$

It is to be mentioned that $\Pr\{Y(t) \leq x\}$ also yield the same limit as $t \to \infty$. This can be observed by noting that if we consider the reversed processes (an identically distributed renewal process), $Y(t)$, the "excess life time" at t is same as the age at t, $A(t)$ of the original process.

Another way of evaluating $\lim_{t\to\infty} \Pr\{A(t) \leq x\}$ is to note that $\{A(t) \leq x\} = \{S_{N(t)} \geq t-x\}$

from which it follows that

$$\begin{aligned}
\Pr\{A(t) \leq x\} &= \Pr\{S_{N(t)} \geq t - x\} \\
&= \int_{-\infty}^{\infty} \Pr\{S_{N(t)} \geq t - x | S_{N(t)} = y\} dF_{S_{N(t)}}(y) \\
&= \int_{t-x}^{\infty} dF_{S_{N(t)}}(y) \\
&\stackrel{(a)}{=} \int_{-\infty}^{x} \bar{F}(u) dm(u) \\
&= \int_{-\infty}^{0} dm(u) + \int_{0}^{x} \bar{F}(u) dm(u) \\
&= \int_{0}^{x} \bar{F}(u) dm(u),
\end{aligned}$$

where (a) follows from a change of variable $u = t - y$. In the limit, $dm(u) \to \frac{du}{\mu}$, as $t \to \infty$, and hence

$$\lim_{t \to \infty} \Pr\{A(t) \leq x\} = \frac{1}{\mu} \int_{0}^{x} \bar{F}(u) du \tag{3}$$