Lecture 07: Inspection Paradox and Limiting Mean Excess Time

1 The Inspection Paradox

Define $X_{N(t)+1} = A(t) + Y(t)$ as the length of the renewal interval containing t, in other words, the length of current renewal interval. Inspection paradox says that $P(X_{N(t)+1} > x) \geq \bar{F}(x)$. That is, for any x, the length of the current renewal interval to be greater than x is always more likely than that for an ordinary renewal interval. Formally,

$$\Pr\{X_{N(t)+1} > x\} = \int_0^t \Pr\{X_{N(t)+1} > x | S_{N(t)} = y, N(t) = n\} dF_{(S_{N(t)}, N(t))}.$$

Now we have,

$$\begin{aligned}
\Pr\{X_{N(t)+1} > x | S_{N(t)} = y, N(t) = n\} &= \Pr\{X_{N(t)+1} > x | X_1 + \cdots + X_n = y, X_{n+1} > t - y\} \\
&= \Pr\{X_{n+1} > x | X_{n+1} > t - y\} \\
&= \frac{\Pr\{X_{n+1} > \max(x, t-y)\}}{\Pr\{X_{n+1} > t - y\}} \\
&\geq \bar{F}(x).
\end{aligned}$$

So we get that,

$$\Pr\{X_{N(t)+1} > x\} \geq \Pr\{X_1 > x\}.$$

One can also look into a weaker version of inspection paradox involving the limiting distribution of $X_{N(t)+1}$, consider an alternating renewal process for which the ON time is the total time of the cycle if that total time is greater than x, and zero otherwise. The system is either totally ON during a cycle (if the renewal interval is greater than x), or totally OFF otherwise. Formally,

$$\begin{aligned}
Z_n &= \text{ON time in } n^{th} \text{ cycle} = X_n \mathbb{I}_{X_n > x} \\
Y_n &= \text{OFF time in } n^{th} \text{ cycle} = X_n \mathbb{I}_{X_n \leq x}.
\end{aligned}$$

Now we have,

$$\begin{aligned}
\Pr\{X_{N(t)+1} > x\} &= \Pr\{\text{length of the interval containing } t > x\} \\
&= \Pr\{\text{on at time } t\}.
\end{aligned}$$

In view of Corollary **??**, we conclude that

$$\begin{aligned}\lim_{t\to\infty}\Pr\{X_{N(t)+1} > x\} &= \frac{\mathbb{E}[\text{on time in cycle}]}{\mu}\\ &= \frac{\mathbb{E}[X\mathbb{I}_{X>x}]}{\mu}\\ &= \frac{\int_x^\infty ydF(y)}{\mu}\\ &\geq \Pr[X_1 \geq x],\end{aligned}$$

where the last step follows from Chebyshev's inequality stated below.

Chebyshev's Sum Inequality:

If $f:\mathbb{R}\to\mathbb{R}^+$ and $g:\mathbb{R}\to\mathbb{R}^+$ are functions with the same monotonicity then for any random variable X, $f(X)$ and $g(X)$ are positive and

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)].$$

Remark:
This inequality gives us that

$$\mathbb{E}[X\mathbb{I}_{X\geq x}] \geq \mathbb{E}[X]\Pr[X \geq x].$$

1.1 Example:

Suppose the number of commodities desired by a customer at a store follows a distribution G. The ordering policy of the store is as follows: For some fixed s, S, if the inventory level after serving a customer is x, then the amount ordered is

$$\begin{cases} S-x & \text{if } x < s\\ 0 & \text{if } x \geq s\end{cases}$$

Let $L(t)$ denote the inventory level at time t. We are interested in finding $\lim_{t\to\infty}\mathbb{P}(L(t)\geq y)$. Let X_n denote inter-restocking times. Let $\{L(t)\geq y\}$ denote ON period. X_n forms an alternating renewal process with the above mentioned ON time. From alternating renewal process theorem, we have

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}(L(t)\geq y) &= \frac{\mathbb{E}[\text{ON time}]}{\mathbb{E}[X_1]}\\ &= \frac{\mathbb{E}[\sum_{i=1}^{N_y} X_i]}{\mathbb{E}[\sum_{i=1}^{N_s} X_i]} = \frac{\mathbb{E}[N_x]}{\mathbb{E}[N_s]}.\end{aligned}$$

where $N_y = \min\{n\in\mathbb{N} : \sum_{i=1}^n D_i > S-y\}$ and $D_1, D_2\ldots$ denote the successive customer demands. Since D_i are iid, we can interpret $N_y - 1$ as the number of renewals till time $S-y$. D_i is the inter arrival time of the process. Thus

$$\lim_{t\to\infty}\mathbb{P}(L(t)\geq y) = \frac{m_G(S-x)+1}{m_G(S-s)+1}, s\leq x\leq S.$$

2 Limiting Mean Excess Time

Consider a nonlattice renewal process and we are interested in computing the mean excess time of the process. We start by writing the renewal equation of mean excess life time, $\mathbb{E}[Y(t)]$.

$$\begin{aligned}\mathbb{E}[Y(t)] &= \mathbb{E}[Y(t)|S_{N(t)} = 0]F^c(t) + \int_0^t \mathbb{E}[Y(t)|S_{N(t)} = y]F^c(t-y)dm(y) \\ &= \mathbb{E}[X_1 - t|X_1 > t]F^c(t) + \int_0^t \mathbb{E}[X - (t-y)|X > t-y]F^c(t-y)dm(y).\end{aligned}$$

From Key Renewal theorem, we have

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}[Y(t)] &= \frac{1}{\mu}\int_0^\infty \mathbb{E}[X - t|X - t > 0]F^c(t)dt \\ &= \frac{1}{\mu}\int_{t=0}^\infty \int_{x=t}^\infty (x-t)dF(x)dt \\ &= \frac{1}{\mu}\int_{x=0}^\infty \int_{t=0}^x (x-t)dF(x)dt \\ &= \frac{\mathbb{E}[X^2]}{2\mu}.\end{aligned}$$

Proposition 2.1. *If the inter arrival time is nonlattice and $\mathbb{E}[X^2] < \infty$, by corollary , we have* $\mu(m(t)+1) = t + \mathbb{E}[Y(t)]$

$$\lim_{t\to\infty}(m(t) - \frac{t}{\mu}) = \frac{\mathbb{E}[X^2]}{2\mu^2} - 1.$$