Lecture 08: Branching Processes and Delayed Renewal Process

1 Age-dependent Branching Process

Suppose an organism lives upto a time period of $X \sim F$ and produces $N \sim P$ number of offspring. Let $X(t)$ denote the number of organisms alive at time t. The stochastic process $\{X(t),\ t \geq 0\}$ is called an age-dependent branching process. We are interested in computing $M(t) = \mathbb{E}[X(t)]$ when $m = \mathbb{E}[N] = \sum_{j \in \mathbb{N}} jP_j$.

Theorem 1.1. *If $X(0) = 1$, $m > 1$ and F is non lattice, then*

$$\lim_{t \to \infty} e^{-\alpha t} M(t) = \frac{m-1}{m^2 \alpha \int_0^\infty x e^{-\alpha x dF(x)}},$$

where $\alpha > 0$ is unique such that $\int_0^\infty x e^{-\alpha x} dF(x) = \frac{1}{m}$.

Proof. Condition on T_1, the life time of first organism,

$$\begin{aligned} M(t) &= \int_0^\infty \mathbb{E}[X(t) | T_1 = y] dF(y) \\ &\overset{(a)}{=} \int_{y=0}^t 1 dF(y) + \int_{y=t}^\infty m M(t-y) dF(y). \end{aligned}$$

Thus we get

$$M(t) = F^c(t) + m \int_0^t M(t-y) dF(y) \tag{1}$$

Let α denote the unique positive number such that $\int_0^\infty x e^{-\alpha x} dF(x) = \frac{1}{m}$ and $G(y) = m \int_0^y e^{-alphay} dF(y)$. Upon multiplying both sides of equation (**??**) by $e^{-\alpha t}$ and defining $f(t) = e^{-\alpha t} M(t)$, $h(t) = e^{-\alpha t} F^c(t)$,

$$\begin{aligned} f &= h + f * G \\ &= h + G * (h + f * G) \\ \vdots &= h + h * \sum_{i=1}^\infty G_i \\ &= h + h * m_G. \end{aligned}$$

Or, $f(t) = h(t) + \int_0^t h(t-s)dm_G(s)$. It can be shown that $h(t)$ is dRi and hence by Key Renewal thmrem,

$$f(t) \to \frac{\int_o^\infty e^{-\alpha t} F^c(t)dt}{\int_0^\infty xdG(x)}.$$

$$\begin{aligned}
\int_o^\infty e^{-\alpha t} F^c(t)dt &= \int_0^\infty e^{-\alpha t} \int_t^\infty dF(x)dt \\
&= \int_0^\infty \int_0^x e^{-\alpha t} dt dF(x) \\
&= \int_0^\infty (1-e^{-\alpha x})dF(x) \\
&= \frac{1}{\alpha}(1-\frac{1}{m}) \text{ (by the definition of } \alpha).
\end{aligned}$$

Also $\int_0^\infty xdG(x) = m\int_0^\infty xe^{-\alpha x}dF(x)$. Hence the result follows.

□

2 Delayed Renewal Process

Let $\{X_n : n \in \mathbb{N}\}$ be independent but $X_1 \sim G$ and $X_i \sim F$, $i \geq 2$ then the counting process $\{N_D(t) : t \geq 0\}$ is called general renewal process or delayed renewal process. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. We have

$$\begin{aligned}
N_D(t) &= \sup\{n \in \mathbb{N} : S_n \leq t\}, \\
P(N_D(t) = n) &= P(S_n \leq t) - P(S_{n+1} \leq t) \\
&= G * F^{n-1}(t) - G * F^n(t), \\
m_D(t) = \mathbb{E}[N_D(t)] &= \sum_{n \in \mathbb{N}} G * F^{n-1}(t).
\end{aligned}$$

Taking the Laplace transform of $m_D(t)$, denoted as $\tilde{m}_D(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)}$.

Proposition 2.1. *The following holds:*

1. $\lim_{t\to\infty} \frac{N_D(t)}{t} = \frac{1}{\mu}$.
2. $\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mu}$.
3. *If F is non-lattice,* $lim_{t\to\infty} m_D(t+a) - m_D(t) = \frac{a}{\mu_F}$.
4. *If F and G are lattice with period d,* $\mathbb{E}[\ \#$*of renewals at nd* $] = \frac{d}{\mu_F}$.
5. *If F is nonlattice, $\mu < \infty$ and h dRi, then*

$$\lim_{t\to\infty} \int_0^t h(t-x)dm_D(x) = \frac{\int_0^\infty h(t)dt}{\mu}.$$

2.0.1 Example:

Let $\{X_n : n \in \mathbb{N}\}$ be iid discrete observed. A pattern $x_1, x_2 \ldots x_k$ is said to occur at time n if $X_n = x_k,\ X_{n-1} = x_{k-1}, \ldots X_{n-k+1} = x_1$.. If we have iid tosses and consider $N(n)$ as the number of times pattern $0, 1, 0, 1$ appear in n tosses, with $P(H) = p = 1 - q$, the process is a delayed renewal processes. To find the mean number of tosses for the first time the pattern $0, 1, 0, 1$ appear,

$$\begin{aligned}\mathbb{E}[\text{first time pattern } 0,1,0,1 \text{ appears}] &= \mathbb{E}[\text{first time pattern } 0,1 \text{ appears}] \\ &\quad + \mathbb{E}[\text{time between patterns } 0,1,0,1] \\ &= p^{-1}q^{-1} + p^{-2}q^{-2}.\end{aligned}$$

Similarly we can show that $\mathbb{E}[\text{first time } k\text{heads}] = \sum_{i=1}^{n} p^{-i}$.