Lecture 09: Equilibrium Renewal Processes and Renewal Reward Processes

1 Renewal theory Contd. – Delayed Renewal processes

1.1 Example:

(Optional – not covered in class)

Consider two coins and suppose that each time is coin flipped, it lands tail with some unknown probability $p_i,\ i = 1, 2$. We are interested in coming up with a strategy that ensures that long term proportion of tails is $\min\{p_1,\ p_2\}$. One strategy is as follows: Set $n = 1$. In the n^{th} round of coin flipping, flip the first coin till n consecutive tails are obtained. Then flip the second coin till n consecutive tails are obtained. Increment n and repeat.

Claim. $\lim_{m\to\infty} \frac{\#\text{tails in the first } m \text{ tosses}}{m} = \min\{p_1, p_2\}$ with probability 1.

The proof is as follows. Let $p = \max\{p_1, p_2\}$ and $\alpha p = \min\{p_1, p_2\}$. There is nothing to prove if $\alpha = 1$, so let $\alpha < 1$. Call the coin with $P(T) = p$, the bad coin and the other, the good coin. Let B_n denote the number of flips in the n^{th} round of tossing the bad coin, and G_n the number of flips in the n^{th} round of tossing the good coin. We first prove the following lemma.

Lemma 1.1. *For any $\epsilon > 0$ with $\epsilon^{-1} \in \mathbb{N}$, $P(B_n \geq \epsilon G_n$ for infinitely many rounds $n) = 0$.*

Proof. For any $n \in \mathbb{N}$,

$$\begin{aligned}
P\left(G_n \leq \frac{B_n}{\epsilon}\right) &= \mathbb{E}[P(G_n \leq \frac{B_n}{\epsilon}|B_n)] \\
&= \mathbb{E}[\sum_{i=1}^{\frac{B_n}{\epsilon}} P(G_n = i|B_n)] \\
&\leq \mathbb{E}[\sum_{i=1}^{\frac{B_n}{\epsilon}} (\alpha p)^n] \\
&= \mathbb{E}[\frac{B_n}{\epsilon}](\alpha p)^n \\
&= \epsilon^{-1}\left(\sum_{i=1}^{n} \frac{1}{p^i}\right)(\alpha p)^n = \epsilon^{-1}\frac{p^{-n}-1}{1-p}(\alpha p)^n,
\end{aligned}$$

where the inequality follows from the fact that $\{G_m = i\}$ implies that $i \geq m$ and that in cycle m, the coin flips numbered $i - m + 1$ to i are all tails. Hence,

$$\sum_{n=1}^{\infty} P\left(G_n \leq \frac{B_n}{\epsilon}\right) \leq \epsilon^{-1}\sum_{n=1}^{\infty}\frac{\alpha^n}{1-p} < \infty.$$

By the Borel-Cantelli lemma, it follows that $P(B_n \geq \epsilon G_n \text{for infinitely many } n) = 0$. □

With probability 1, all but a finite number of rounds have at most an ϵ fraction of bad coin tosses, implying that $\lim_{m\to\infty} \frac{\#\text{bad coin tosses in the first } m \text{ tosses}}{m} \leq \epsilon$. Now taking a decreasing sequence $\epsilon_k = 1/k$, $k = 1, 2, 3, \ldots$, and using the continuity of probability, we get that with probability 1, $\lim_{m\to\infty} \frac{\#\text{bad coin tosses in the first } m \text{ tosses}}{m} = 0$. This proves the claim using the strong law of large numbers for tosses of the good coin.

1.2 Distribution of the Last Renewal Time for a Delayed Renewal Process

In the same manner as we derived the key lemma, refer Theorem 1.9 in lecture 6, for the last renewal time distribution of a standard renewal process, we can show for a delayed renewal process:

$$\begin{aligned} P(S_{N(t)} \leq s) &= G^c(t) P(S_{N(t)} \leq s | S_{N(t)} = 0) + \int_0^t P(S_{N(t)} \leq s | S_{N(t)} = u) F^c(t-u) dm(u) \\ &= G^c(t) + \int_0^s F^c(t-u) dm(u). \end{aligned}$$

Let $F_e(x) = \frac{\int_0^x F^c(y)dy}{\mu}$, $x \geq 0$, known as the *equilibrium distribution* of F. Observe that the moment generating function of $F_e(x)$ is $\tilde{F}_e(s) = \frac{1-\tilde{F}(s)}{s\mu}$.

Proof. By definition, $\tilde{F}_e(s) = \mathbb{E}\left[\mathrm{e}^{-sX}\right]$, where X is a random variable with probability distribution function $F_e(x)$. So,

$$\begin{aligned} \tilde{F}_e(s) &= \int_0^\infty \mathrm{e}^{-sx} dF_e(x) \\ &= \frac{1}{\mu} \int_0^\infty \mathrm{e}^{-sx} F^c(x) dx \\ &= \frac{1}{s\mu} - \frac{1}{\mu} \int_0^\infty \mathrm{e}^{-sx} F(x) dx \\ &= \frac{1}{s\mu} - \frac{1}{s\mu} \int_0^\infty \mathrm{e}^{-sx} dF(x) \\ &= \frac{1}{s\mu} - \frac{1}{s\mu} \tilde{F}(s), \end{aligned}$$

where the third and fourth equalities follows from the basic integration techniques. □

And also observe that F_e is the limiting distribution of the age and the excess time for the renewal process governed by F. If $G = F_e$, then the delayed renewal process is called the *equilibrium renewal process.* Suppose we start observing a renewal process at some arbitrary time t. Then, the observed renewal process is the equilibrium renewal process. Let $Y_e(t)$ denote the excess time for the (delayed) equilibrium renewal process.

Theorem 1.2. *For the equilibrium renewal process,*

1. $m_e(t) = \frac{t}{\mu}$.

2. $P(Y_e(t) \le x) = F_e(x)$.

3. $\{N_e(t), t \ge 0\}$ *has stationary increments.*

Proof. To prove (1), observe that $\tilde{m}_e(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)} = \frac{\tilde{F}_e(s)}{1-\tilde{F}(s)} = \frac{1}{s\mu}$. Hence, if $m_e(t) = \frac{t}{\mu}$ then,

$$\begin{aligned}\tilde{m}_e(s) &= \int_0^\infty \mathrm{e}^{-st} dm_e(t) \\ &= \frac{1}{\mu}\int_0^\infty \mathrm{e}^{-st} dt \\ &= \frac{1}{s\mu}.\end{aligned}$$

Since moment generating function is a one-to-one map, $m_e(t) = \frac{t}{\mu}$ is unique.
(2)

$$\begin{aligned}P(Y_e(t) > x) &= P(Y_e(t) > x | S_{N_e(t)} = 0) P(S_{N_e(t)} = 0) + P(Y_e(t) > x | S_{N_e(t)} = s) F^c(t-s)\frac{ds}{\mu} \\ &= P(X_1 > t + x, X_1 > t) + P(X_2 > t + x - s | X_2 > t - s) F^c(t-s)\frac{ds}{\mu} \\ &= {F_e}^c(t+x) + \int_0^t F^c(t+x-s)\frac{ds}{\mu} \\ &= 1 - \frac{1}{\mu}\int_0^{t+x} F^c(y)dy - \frac{1}{\mu}\int_{t+x}^x F^c(y)dy \\ &= 1 - \frac{1}{\mu}\int_0^x F^c(y)dy \\ &= F_e^c(x).\end{aligned}$$

(3) $N_e(t+s) - N_e(s) =$ Number of renewals in time interval of length t. When we start observing at s, the observed renewal process is delayed renewal process with initial distribution being the original distribution. □

Question: What can you say about the equilibrium renewal process when F is distributed exponentially with the parameter λ?
Answer: Let's look at the distribution of the first inter-arrival distribution, F_e. So,

$$\begin{aligned}F_e(x) &= \frac{1}{\mu}\int_0^x F^c(y)dy \\ &= \lambda\int_0^x \mathrm{e}^{-y\lambda}dy \\ &= 1 - \mathrm{e}^{-x\lambda},\end{aligned}$$

where the first equality follows from the definition of F_e for equilibrium renewal process, the second equality follows from the fact that the mean of exponential distribution is inverse of the parameter λ.
Thus even F_e is distributed exponentially with the parameter λ. So with all the properties of equilibrium renewal process, F_e and F being distributed exponentially with the same parameter λ, says that this is a poisson process (not a delayed renewal process).

1.3 Renewal Reward Process

Definition: Consider a renewal process $\{N(t), t \geq 0\}$ with inter arrival times $\{X_n : n \in \mathbb{N}\}$ having distribution F and rewards $\{R_n : n \in \mathbb{N}\}$ where R_n is the reward at the end of X_n. Let (X_n, R_n) be iid. Then $R(t) = \sum_{i=1}^{N(t)} R_i$ is reward process (total reward earned by time t).

Theorem 1.3. *Let $\mathbb{E}[|R|]$ and $\mathbb{E}[|X|]$ be finite.*

1. $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$ *a.s.*
2. $\lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$.

Proof. (1) Write

$$\begin{aligned}\frac{R(t)}{t} &= \frac{\sum_{i=1}^{N(t)} R_i}{t} \\ &= \left(\frac{\sum_{i=1}^{N(t)} R_i}{N(t)}\right)\left(\frac{N(t)}{t}\right).\end{aligned}$$

By the strong law of large numbers (almost sure convergence law) we obtain that,

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} = \mathbb{E}[R],$$

and by the basic renewal theorem (almost sure convergence law) we obtain that,

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X]}.$$

Thus (1) is proven.
(2)
Notice that $N(t) + 1$ is a stopping time for the sequence $\{R_1, R_2, \ldots\}$. This is true since

$$\begin{aligned}\{N(t) + 1 = n\} &= \{X_1 + X_2 + \cdots + X_{n-1} \leq t, X_n > t\} \\ &= \{R_1 + R_2 + \cdots + R_{n-1} = R(t), R_n \neq 0\}.\end{aligned}$$

Moreover $N(t) + 1$ is a stopping time for the sequence $\{X_1, X_2, \ldots\}$. So by algebra and Wald's lemma,

$$\begin{aligned}\mathbb{E}[R(t)] &= \mathbb{E}\left[\sum_{i=1}^{N(t)} R_i\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{N(t)+1} R_i\right] - \mathbb{E}[R_{N(t)+1}] \\ &= (m(t) + 1)\mathbb{E}[R_1] - \mathbb{E}[R_{N(t)+1}].\end{aligned}$$

Let $g(t) = \mathbb{E}[R_{N(t)+1}]$. So

$$\frac{\mathbb{E}[R(t)]}{t} = \frac{(m(t) + 1)}{t}\mathbb{E}[R_1] - \frac{g(t)}{t},$$

and the result will follow from the elementary renewal theorem if we can show that $\frac{g(t)}{t} \to 0$ as $t \to \infty$. So,

$$\begin{aligned} g(t) &= \mathbb{E}[R_{N(t)+1} 1\{S_{N(t)} = 0\}] + \mathbb{E}[R_{N(t)+1} 1\{S_{N(t)} > 0\}] \\ &= \mathbb{E}[R_{N(t)+1} | S_{N(t)} = 0] P(X_1 > t) + \int_0^t \mathbb{E}[R_{N(t)+1} | S_{N(t)} = u] F^c(t-u) dm(u), \end{aligned}$$

where the second equality follows from the fact that the interarrival times $X_n, n \in \mathbb{N}$, are <u>iid</u> with distribution F.
However,

$$\begin{aligned} \mathbb{E}[R_{N(t)+1} | S_{N(t)} = 0] = \mathbb{E}[R_1 | X_1 > t], \\ \mathbb{E}[R_{N(t)+1} | S_{N(t)} = u] = \mathbb{E}[R_n | X_1 > t-u], \end{aligned}$$

and so

$$\begin{aligned} g(t) &= \mathbb{E}[R_1 | X_1 > t] F^c(t) + \int_0^t \mathbb{E}[R_n | X_1 > t-u] F^c(t-u) dm(u) \\ &= \mathbb{E}[R_1 | X_1 > t] F^c(t) + \int_0^t \mathbb{E}[R_1 | X_1 > t-u] F^c(t-u) dm(u), \end{aligned}$$

where the second equality follows from the fact that $R_n, n \in \mathbb{N}$, are <u>iid</u>.
Now, let

$$h(t) = \mathbb{E}[R_1 | X_1 > t] F^c(t) = \int_{x=t}^{\infty} \mathbb{E}[R_1 | X_1 = x] dF(x),$$

and note that since

$$\mathbb{E}[|R_1|] = \int_{x=0}^{\infty} \mathbb{E}[|R_1| | X_1 = x] dF(x) < \infty,$$

it follows that $h(t) \to 0$ as $t \to \infty$. Hence, choosing T such that $|h(u)| \le \epsilon$ whenever $u \ge T$, we have for all $t \ge T$ that

$$\begin{aligned} \frac{|g(t)|}{t} &\le \frac{|h(t)|}{t} + \int_0^{t-T} \frac{|h(t-s)|}{t} dm(s) + \int_{t-T}^t \frac{|h(t-s)|}{t} dm(s) \\ &\le \frac{\epsilon}{t} + \frac{\epsilon m(t-T)}{t} + \mathbb{E}[|R_1|] \frac{(m(t) - m(t-T))}{t}. \end{aligned}$$

Hence $\lim_{t\to\infty} \frac{g(t)}{t} = \frac{\epsilon}{\mathbb{E}[X]}$ by the elementary renewal theorem, and the result follows since $\epsilon > 0$ is arbitrary. □

Remark 1.4. (1) $R_{N(t)+1}$ has different distribution than R_1.
<u>Analysis:</u> Notice that $R_{N(t)+1}$ is related to $X_{N(t)+1}$ which is the length of the renewal interval containing the point t. Since larger renewal intervals have a greater chance of containing t, it follows that $X_{N(t)+1}$ tends to be larger than a ordinary renewal interval. Formally,

$$\Pr\{X_{N(t)+1} > x\} = \sum_{n \in \mathbb{N}_0} \left(\left[\int_0^t \Pr\{X_{N(t)+1} > x | S_{N(t)} = y, N(t) = n\} F^c(t-y) dm(y) \right] \Pr\{N(t) = n\} \right).$$

Now we have,

$$\begin{aligned}\Pr\{X_{N(t)+1} > x | S_{N(t)} = y, N(t) = n\} &= \Pr\{X_{N(t)+1} > x | X_1 + \cdots + X_n = y, X_{n+1} > t - y\} \\ &= \Pr\{X_{n+1} > x | X_{n+1} > t - y\} \\ &= \frac{\Pr\{X_{n+1} > \max(x, t-y)\}}{\Pr\{X_{n+1} > t - y\}} \\ &\geq F^c(x).\end{aligned}$$

So we get that,

$$\Pr\{X_{N(t)+1} > x\} \geq \Pr\{X_1 > x\}.$$

Thus the remark follows.
(2) $R(t)$ is the gradual reward during a cycle,

$$\frac{\sum_{n=1}^{N(t)} R_n}{t} \leq \frac{R(t)}{t} \leq \frac{\sum_{n=1}^{N(t)+1} R_n}{t}.$$

Analysis: The part 1 of the theorem 1.3 under this regime follows since

$$\lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]},$$
$$\lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)+1} R_n}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]},$$

by the similar arguments given in the proof of the theorem 1.3.
The part 2 of the theorem 1.3 under this regime follows since

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[R_{N(t)+1}\right]}{t} = 0,$$

by the similar arguments given in the proof of the theorem 1.3. Thus the remark follows. For more insights refer Chapter 3 in *Stochastic Processes* by *Sheldon M. Ross.*

1.3.1 Example:

Suppose for an alternating renewal process, we earn at a rate of one per unit time when the system is on and the reward for a cycle is the the time system is ON during that cycle.
$\lim_{t\to\infty} \frac{\text{Amount of ON time in [0,t]}}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[X]}{\mathbb{E}[X]+\mathbb{E}[Y]} = \lim_{t\to\infty} P(\text{ON at time t}).$