Lecture 11 : Discrete Time Markov Chains

1 Discrete Time Markov Chains Contd

Theorem 1.1. *An irreducible, aperiodic Markov Chain with countable state space I is of one of the following types:*

i) All the states are either transient or null recurrent. That is, $\lim_{n\in\mathbb{N}} P^n_{ij} = 0$ and there exists no stationary distribution.

ii) All the states are positive recurrent. There exists a unique stationary distribution $\pi \in \Delta(I)$,

$$\pi_j = \lim_{n\in\mathbb{N}} P^n_{ij} > 0,\ j \in I.$$

Proof. Let $\{X_n : n \in \mathbb{N}\}$ be an irreducible, aperiodic Markov chain with state space I.

i) If the states are transient or null recurrent and $P \in \Delta(I)$ is a stationary distribution, then for any $n \in \mathbb{N}$, we have

$$P_j = \sum_{i\in I} \Pr\{X_n = j|X_0 = i\} \Pr\{X_0 = i\} = \sum_{i\in I} \pi_i P^n_{ij}.$$

Since, $\pi_j = 0$ for all $j \in I$, we have $P_j = 0$ for every $j \in I$. This contradicts the assumption that P_j is a probability distribution.

ii) From renewal reward theorem, $\lim_{n\in\mathbb{N}} P^{(n)}_{ij} = 1/\mu_{jj}$ exists, which we take as π_j. Further, for any finite set $A \subseteq I$, we have

$$\sum_{j\in A} P^{(n)}_{ij} \leq \sum_{j\in I} P^{(n)}_{ij} = 1.$$

Taking limit $n \in \mathbb{N}$ on both sides, we conclude that $\sum_{j\in A} \pi_j \leq 1$ for all A finite. Taking limit with respect to A, we conclude,

$$\sum_{j\in I} \pi_j \leq 1.$$

Further, we can write for all $A \subseteq I$,

$$P^{n+1}_{ij} = \sum_{k\in I} P^n_{ik} P_{kj} \geq \sum_{k\in A} P^n_{ik} P_{kj}.$$

Applying limit $n \in \mathbb{N}$ on both sides, we get $\pi_j \geq \sum_{k\in A} \pi_k P_{kj}$ for all A finite. Hence, taking limits with respect to A, we get for all $j \in I$,

$$\pi_j \geq \sum_{k\in I} \pi_k P_{kj}.$$

Assuming that the inequality is strict for some $j \in I$, we can sum the inequalities over j. Since, summands are non-negative we can exchange summation orders to get

$$\sum_{j\in I} \pi_j > \sum_{j\in I}\sum_{k\in I} \pi_k P_{kj} = \sum_{k\in I} \pi_k \sum_{j\in I} P_{kj} = \sum_{k\in I} \pi_k.$$

This is a contradiction. Therefore, for all $j \in I$

$$\pi_j = \sum_{k\in I} \pi_k P_{kj}.$$

Defining normalized $P_j = \frac{\pi_j}{\sum_{k\in I}\pi_k}$, we see that $\{P_j, j \in I\}$ is a stationary distribution and so at least one stationary distribution exists. Let $\{P_j, j \in I\}$ be any stationary distribution. Let $\{P_j, j \in I\}$ be the probability distribution of X_0, then for any finite subset $A \subseteq I$, we get

$$P_j = \Pr\{X_n = j\} = \sum_{i\in I} \Pr\{X_n = j | X_0 = i\} P\{X_0 = i\} = \sum_{i=0}^{\infty} P_{ij}^n P_i \geq \sum_{i\in A} P_{ij}^n P_i.$$

As before, we take limit $n \in \mathbb{N}$, followed by limit of increasing subsets $A \uparrow I$, to obtain

$$P_j \geq \sum_{i\in I} \pi_j P_i = \pi_j.$$

To show $P_j \leq \pi_j$, we use the fact that $P_{ij}^n \leq 1$. Let $A \subseteq I$ be a finite subset, then

$$P_j = \sum_{i\in I} P_{ij}^n P_i = \sum_{i\in A} P_{ij}^n P_i + \sum_{i\in A^c} P_{ij}^n P_i \leq \sum_{i\in A} P_{ij}^n P_i + \sum_{i\in A^c} P_i.$$

Using our standard approach of taking limit $n \in \mathbb{N}$, followed by $A \subseteq I$, we obtain

$$P_j \leq \sum_{i=0}^{\infty} \pi_j P_i = \pi_j.$$

□

Corollary 1.2. *An irreducible, aperiodic DTMC defined on a finite state space I will be positive recurrent.*

Proof. Suppose that the DTMC is not positive recurrent, then

$$\lim_{n\in\mathbb{N}} P_{ij}^{(n)} = 0.$$

Interchanging limit and finite summation gives

$$0 = \sum_{j\in I} \lim_{n\in\mathbb{N}} P_{ij}^{(n)} = \lim_{n\in\mathbb{N}} \sum_{j\in I} P_{ij}^{(n)} = 1.$$

This is a contradiction. Hence the DTMC mentioned above is positive recurrent. □

Corollary 1.3. *For an irreducible and aperiodic DTMC with stationary distribution π, we have $\mathbb{E}_i[T_i] = 1/\pi_i$ for any state $i \in I$, where $\mathbb{E}_i[T_i]$ denotes the expected number of time steps to return to the state i.*

1.1 Ergodic theorem for DTMCs

Proposition 1.4. *Let $\{X_n\}$ be an irreducible, aperiodic and positive recurrent DTMC on countable state space I with stationary distribution π. Let $f : I \to R$, such that $\sum_{i\in I} |f(i)|\pi_i < \infty$, that is f is integrable over I with respect to π. Then for any initial distribution of X_0,*

$$\lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=1}^{n} f(X_i) = \sum_{i\in I}\pi_i f(i) \text{ almost surely.}$$

Proof. Fix $X_0 = i \in I$. Let $(\tau_1, \tau_2, \dots)$ be successive instants at which state i is visited, i.e. $\tau_0 = 0$. For all $p \geq 0$, let $R_{p+1} = \sum_{n=\tau_p+1}^{\tau(p+1)} f(X_n)$ be the net reward earned at the end of cycle $(p+1)$. Each cycle forms a renewal. By the strong Markov property, these cycles are independent. At each of these stopping times, Markov chain is in state i. Average reward earned during the first k cycles is given by

$$\frac{1}{n}\sum_{k=1}^{n} R_k = \frac{1}{n}\sum_{k=1}^{n}\sum_{\tau_{k-1}+1}^{\tau_k} f(X_i) = \frac{1}{n}\sum_{t=1}^{\tau_n} f(X_t).$$

From renewal reward theorem and applying previous corollary, we get

$$\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} R_k}{n} = \frac{\mathbb{E}_i[\text{reward in one renewal cycle}]}{\mathbb{E}_i[\text{renewal cycle length}]} = \pi_i \mathbb{E}_i[\sum_{n=1}^{\tau_1} f(X_n)].$$

We focus on the mean reward in one renewal cycle, assuming f is non-negative,

$$\mathbb{E}_i[\text{cycle reward}] = \mathbb{E}_i \sum_{n=1}^{\tau_1}\sum_{j\in I} f(j)1_{\{X_n=j\}} = \sum_{j\in I} f(j)\mathbb{E}_i \sum_{n=1}^{\tau_1} 1_{\{X_n=j\}}.$$

To calculate the reward given above, we need to find the expected number of visits to state j between successive visits to state i. We denote the average number of visits to state j by $Y_j = \lim_{n\in\mathbb{N}} \frac{\sum_{t=1}^{n} 1_{\{X_t=j\}}}{n}$. We can compute Y_j in two ways.

1. DTMC undergoes a renewal each time it hits state j. So, from elementary renewal theorem,

$$Y_j = \mathbb{E}_i 1_{\{X_n=j\}} = 1/(1/\pi_j) = \pi_j.$$

2. Without loss of generality, renewal occurs whenever DTMC hits state i. Hence, $\mathbb{E}_i\tau_1 = 1/\pi - i$ and by renewal reward theorem, we obtain

$$Y_j = \frac{\mathbb{E}_i[\sum_{n=1}^{\tau_1} 1_{\{X_n=j\}}]}{1/\pi_i}.$$

Combining above two, we get $\mathbb{E}_i[\sum_{t=1}^{\tau_1} 1_{\{X_t=j\}}] = \frac{\pi_j}{\pi_i}$ and the result follows. □