# Lecture 13: Foster-Lyapunov Theorem

## 1 Foster's Theorem

**Theorem 1.1 (Foster,1950).** *Let $\{X_n\}_{n\geq 0}$ be a irreducible DTMC on $\mathbb{N}_0$ if there exist a function $L : \mathbb{N}_0 \longrightarrow \mathbb{R}_+$ with $\mathbb{E}[L(X_0)] < \infty$, such that for some $K > k \geq 0$, and $\epsilon > 0$:*

1. $|\{x \in \mathbb{N}_0 : L(x) \leq k\}| < \infty$
2. $\mathbb{E}[L(X_n)|X_{n-1}] < K$*,when* $L(X_{n-1}) \leq k$.
3. $\mathbb{E}[L(X_n) - L(X_{n-1})|X_{n-1}] < -\epsilon$ *if* $L(X_{n-1}) \geq k$

*Then $\{X_n\}_{n\geq 0}$ is positive recurrent.(L $\equiv$ "potential function" or energy or lyapunov function).*

*Proof.* As DTMC is irreducible than enough to show that some state is positive recurrent. By renewal theory, for ant DTMC, for all $x \in \mathbb{N}_0$,

$$\lim_{N\to\infty} \mathbb{E}\Big[\sum_{n=1}^{N} \mathbf{1}_{\{X_n=x\}}\Big] = \frac{1}{\mu_{xx}} \tag{1}$$

where

$$\mu_{xx} = \begin{cases} \infty & if \text{ x } is\ transient \\ \sum_{m\geq 0} m f_{xx}^m & if \text{ x } is\ recurrent \end{cases}$$

consider the RHS of equation (1)

$$\lim_{N\to\infty} \mathbb{E}\Big[\sum_{n=1}^{N} \mathbf{1}_{\{X_n=x\}}\Big] > 0 \Longleftrightarrow x\ is\ positive\ recurrent$$

consider

$$
\begin{aligned}
&0 \le \mathbb{E}[L(X_n)] = \mathbb{E}[L(X_0)] + \sum_{n=1}^{N} \mathbb{E}[L(X_n) - L(X_{n-1})] \\
&= \mathbb{E}[L(X_0)] + \sum_{n=1}^{N} \mathbb{E}[L(X_n) - L(X_{n-1})]\mathbf{1}_{\{L(X_{n-1})>k\}} + \sum_{n=1}^{N} \mathbb{E}[L(X_n) - L(X_{n-1})]\mathbf{1}_{\{L(X_{n-1})\le k\}} \\
&\le \mathbb{E}[L(X_0)] + \sum_{n=1}^{N} \mathbb{E}[-\epsilon\ \mathbf{1}_{\{L(X_{n-1})>k\}}] + \sum_{n=1}^{N} \mathbb{E}[K\ \mathbf{1}_{\{L(X_{n-1})\le k\}}] \\
&\Rightarrow \left[\mathbb{E}\sum_{n=1}^{N} \mathbf{1}_{\{L(X_{n-1})\le k\}}\right](K+\epsilon) \ge -\mathbb{E}[L(X_0)] + \epsilon N \\
&\Rightarrow \frac{1}{N}\left[\mathbb{E}\sum_{n=1}^{N} \mathbf{1}_{\{L(X_{n-1})\le k\}}\right] \ge -\frac{\mathbb{E}[L(X_0)]}{K+\epsilon} + \frac{\epsilon}{K+\epsilon} \\
&\Rightarrow \limsup_{N\to\infty} \frac{1}{N}\left[\mathbb{E}\sum_{n=1}^{N} \mathbf{1}_{\{L(X_{n-1})\le k\}}\right] \ge \frac{\epsilon}{K+\epsilon}
\end{aligned}
$$

Let $F = \{x \in \mathbb{N}_0 : L(x) \le k\},\ |F| < \infty$. Now,

$$
\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N}\left[\mathbb{E}\sum_{n=1}^{N} \mathbf{1}_{\{L(X_{n-1})\le k\}}\right] \\
&= \limsup_{N\to\infty} \frac{1}{N}\left[\mathbb{E}\sum_{n=1}^{N}\sum_{x\in F} \mathbf{1}_{\{X_{n-1}\le k\}}\right] \le \sum_{x\in F}\left[\limsup_{N\to\infty} \frac{1}{N}\mathbb{E}\sum_{n=1}^{N} \mathbf{1}_{\{X_{n-1}\le k\}}\right] \\
&= \sum_{x\in F}\left[\limsup_{N\to\infty} \frac{1}{N}\mathbb{E}\sum_{n=1}^{N} \mathbf{1}_{\{X_{n-1}\le k\}}\right] \ge \frac{\epsilon}{K+\epsilon} > 0
\end{aligned}
$$

Therefore there exist some $x \in F$ such that ,

$$
\limsup_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\sum_{n=1}^{N} \mathbf{1}_{\{X_{n-1}\le k\}}\right] \ge \frac{\epsilon}{(K+\epsilon)|F|} > 0
$$

Therefore there exist some $x \in F$ such that ,

$$
\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\sum_{n=1}^{N} \mathbf{1}_{\{X_{n-1}\le k\}}\right] > 0
$$

□

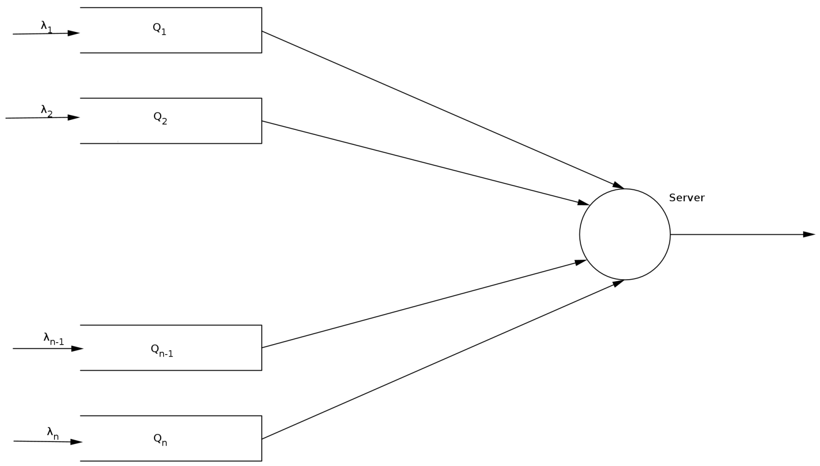


Figure 1: N queues single server, server chooses one queue and serve upto one packet

# 2 Applications of Foster's theorem: Queue scheduling/Max-weight scheduling

Consider $N$ queue served by a single server in discrete time(Figure 1). At time slot t = 1,2,3.... $A_i(t) \in \mathbb{N}_0$ packets arrive to each queue $i \in [N]$ independently.

1. $\mathbb{E}[A_i(t)] = \lambda_i$
2. $\mathbb{P}[A_i(t) = 0] > 0$
3. $\mathbb{E}[A_i(t)^2] \leq C$

Server picks one queue $Q(t) \in [N]$ for service. Let $R_i(t) = \mathbf{1}\{Q(t) = i\}$.One packet is served from $Q(t)$ if it is not empty.Let $X_i(t)$ = number of packets in queue $i$ just before time slot $t$.

$$X_i(t+1) = (X_i(t) + A_i(t) - R_i(t))_+$$

Where

$$a_+ = max(0, a)$$

aslo

$$X_i(t+1) = X_i(t) + A_i(t) - R_i(t) + L_i(t)$$

Where

$$L_i(t) = \begin{cases} 1 & \textit{service attempted when } i \textit{ is empty} \\ 0 & \textit{otherwise} \end{cases}$$

Mean rate of arrivals to system : $= \sum_{i=1}^{N} \lambda_i$
Maximum rate of departure = 1, we will assume $\sum_{i=1}^{N} \lambda_i < 1$.

**Theorem 2.1 (Max-weight scheduling algorithm, 1992).**

$$Q(t) = \arg\max_{i \in N} X_i(t)$$

*that is serve the longest queue.Under MAX-WT ,$X(t) = (X_i(t))_{i=1}^{N}$ is a DTMC which is irreducible and aperiodic on state space $\mathbb{N}_0^N$.As long as $\sum_{i=1}^{N} \lambda_i < 1$, $\{X_n\}$ is positive recurrent.*

*Proof.* By Foster's theorem, define the Lyapunov function:

$$L(x) = \frac{1}{2}\sum_{i=1}^{N} x_i^2$$

consider

$$\begin{aligned} & L(X(t)) - L(X(t-1)) \\ &= \frac{1}{2}\sum_{i=1}^{N} \Big[ (X_i(t))^2 - (X_i(t-1))^2 \Big] \\ &= \frac{1}{2}\sum_{i=1}^{N} \Big[ \{X_i(t-1) + A_i(t-1) - R_i(t-1) + L_i(t-1)\}^2 - (X_i(t-1))^2 \Big] \\ &\le \frac{1}{2}\sum_{i=1}^{N} \Big[ \{X_i(t-1) + A_i(t-1) - R_i(t-1)\}^2 - (X_i(t-1))^2 \Big] \end{aligned}$$

Therefore

$$\mathbb{E}\Big[L(X(t)) - L(X(t-1)) | X(t-1) = x\Big] \le \frac{1}{2}\sum_{i=1}^{N} \mathbb{E}\Big[(x_i + A_i(t-1) - R_i(t))^2 - x_i^2 | X(t-1) = x\Big]$$

$$\begin{aligned} &= \frac{1}{2}\sum_{i=1}^{N} \Bigg[ 2x_i \mathbb{E}\Big[(A_i(t-1) - R_i(t) | X(t-1) = x\Big] + \mathbb{E}\Big[(A_i(t-1) - R_i(t-1))^2 | X(t-1) = x\Big] \Bigg] \\ &= \sum_{i=1}^{N} x_i\lambda_i - \sum_{i=1}^{N} x_i R_i(t-1) + \frac{N}{2}(1+C) \\ &= \sum_{i=1}^{N} x_i\lambda_i + \frac{N}{2}(1+C) - \max_i x_i \\ &\le \frac{N}{2}(1+C) + (\max_i x_i)(\sum_{i=1}^{N} \lambda_i - 1) \\ &= C_1 - \epsilon(\max_i x_i) \end{aligned}$$

Foster's theorem applies with $k = \max\{\frac{\|x\|_2^2}{2}\}$ $\square$