Lecture 14 : Continuous Time Markov Chains

1 Markov Process

Definition 1.1. For a countable set I a continuous time stochastic process $\{X(t) \in I,\ t \geqslant 0\}$ is a **Markov process** if

$$\Pr\{X(t+s) = j | X(u),\ u \in [0,s]\} = \Pr\{X(t+s) = j | X(s)\},\ \text{for all } s,t \geqslant 0 \text{ and } i,j \in I.$$

We define the **transition probability** from state i at time s to state j at time $s+t$ as

$$P_{ij}(s, s+t) = \Pr\{X(s+t) = j | X(s) = i\}.$$

Definition 1.2. The Markov process has **homogeneous** transitions if $P_{ij}(s, s+t) = P_{ij}(0,t)$ for all $i,j \in I, s,t \geqslant 0$ and we denote the transition probability by $P_{ij}(t)$. This continuous Markov process with homogeneous jump transition probabilities is referred to as **continuous time Markov chain (CTMC)**.

Definition 1.3. A distribution π is an **equilibrium distribution** of a Markov process if

$$\pi P(t) = \pi, \quad \forall t \geqslant 0.$$

1.1 Sojourn Times and Jump Transitions

For any stochastic process with countable state space I, we are interested in knowing probabilities of the form $\Pr\{X(v) = i,\ v \in [s, s+t] | X(u) = i,\ u \in [0,s]\}$. To this end, we define sojourn times in any state and and jump transition probabilities from it.

Definition 1.4. Sojourn time of a stochastic process $\{X(t), t \geqslant 0\}$ is defined by

$$\tau_i \triangleq \inf\{t \geqslant 0 : X(t) \neq i | X(0) = i\}, i \in I.$$

Definition 1.5. Jump times of a stochastic process $\{X(t), t \geqslant 0\}$ are defined by

$$S_n \triangleq \inf\{t \geqslant 0 : X(t) \neq X(S_{n-1})\}, \qquad S_0 = 0.$$

We can define a discrete time process $\{X^J(n) : n \in \mathbb{N}_0\}$ called **jump process** from continuous time stochastic process $\{X(t) : t \geqslant 0\}$ by

$$X^J(n) = X(S_n).$$

Definition 1.6. Jump transition probabilities of a stochastic process $\{X(t), t \geqslant 0\}$ are defined by

$$p_{ij}(n) \triangleq \Pr\{X(S_n) = j | X(S_{n-1}) = i\},\ i,j \in I.$$

Lemma 1.7. *For a homogeneous CTMC, sojourn time τ_i is a continuous memoryless random variable.*

Proof. We observe that,

$$\begin{aligned}\Pr\{\tau_i \geqslant s+t|\tau_i > s\} &= \Pr\{X(v) = i,\ v \in [s, s+t)|X(u) = i, i \in [0, s]\} \\ &= \Pr\{X(v) = i,\ v \in [0, t)|X(0) = i\} = \Pr\{\tau_i \geqslant t\}.\end{aligned}$$

□

Lemma 1.8. *For any stochastic process, jump transition probabilities $p_{ij}(\tau_i)$ add to unity for all $i \in I$.*

Proof. It follows from simple law of total probability. □

1.2 Alternative construction of CTMC

Proposition 1.9. *A stochastic process $\{X(t) \in I, t \geqslant 0\}$ is a CTMC iff*

a. sojourn times are independent and identically distributed with an exponential distribution with rate ν_i, and

b. jump transition probabilities $p_{ij}(\tau_i)$ are independent of τ_i, such that $\sum_{i \neq j} p_{ij} = 1$.

Proof. Necessity of two conditions follows from Lemma 1.7 and 1.8. For sufficiency, we assume both conditions and show that Markov property holds and the transition probability is homogeneous. To this end, we observe

$$\{X(u) = i, u \in [0, s]\} = \{\tau_i > s\}.$$

Hence, we can write for small $t > 0$,

$$\Pr\{X(t+s) = j|X(u) = i, u \in [0, s]\} = \int_0^t \frac{\Pr\{\tau_i = s+u\}\Pr\{\tau_j > t-u\}}{\Pr\{\tau_i > s\}} p_{ij} + o(t).$$

Since τ_i's are independent and memoryless, it follows that

$$\Pr\{X(t+s) = j|X(u) = i, u \in [0, s]\} = \Pr\{X(t) = j|X(0) = i\}, \quad \forall s, t \geqslant 0.$$

□

Remark 1.10. Transition probabilities p_{ij} and sojourn times τ_i are independent.

Remark 1.11. Inverse of mean sojourn time ν_i is called rate of state i, and typically $\nu_i < \infty$.

Remark 1.12. If $\nu_i = \infty$, we call the state to be instantaneous.

Remark 1.13. A CTMC is a DTMC with exponential sojourn time in each state.

Definition 1.14. A CTMC is called **regular** if

$$\Pr\{\text{ number of transitions in } [0, t] \text{ is finite}\} = 1,\ \forall t < \infty.$$

Example 1.15. Consider the following example of a non-regular CTMC. $p_{i,i+1} = 1, \nu_i = i^2$. Show that it is not regular.

2 Generator Matrix

Definition 2.1. A **generator matrix** denoted by $Q \in \mathbb{R}^{I \times I}$ is defined in terms of sojourn times $\{\nu_i, i \in I\}$ and jump transition probabilities $\{p_{ij}, i \neq j \in I\}$ of a CTMC as

1. $q_{ii} = -\nu_i$,
2. $q_{ij} = \nu_i p_{ij}$.

Lemma 2.2. *A matrix Q is generator matrix for a CTMC iff for each $i \in I$,*

1. $0 \leq -q_{ii} < \infty$,
2. $q_{ij} \geq 0$,
3. $\sum_{j \in I} q_{ij} = 0$.

From the Q matrix, we can construct the whole CTMC. In DTMC, we had the result $P^{(n)}(i,j) = (P^n)_{i,j}$. We can generalize this notion in the case of CTMC as follows: $P = e^Q \triangleq \sum_{k \in \mathbb{N}_0} \frac{Q^k}{k!}$. Observe that $e^{Q_1+Q_2} = e^{Q_1}e^{Q_2},\ e^{nQ} = (e^Q)^n = P^n$.

Theorem 2.3. *Let Q be a finite sized matrix. Let $P(t) = e^{tQ}$. Then $\{P(t),\ t \geq 0\}$ has the following properties:*

1. $P(s+t) = P(s)P(t),\ \forall s,\ t$ *(semi group property).*
2. $P(t),\ t \geq 0$ *is the unique solution to the forward equation,* $\frac{dP(t)}{dt} = P(t)Q,\ P(0) = I$.
3. *And the backward equation* $\frac{dP(t)}{dt} = QP(t),\ P(0) = I$.
4. *For all* $k \in \mathbb{N}$, $\frac{d^k P(t)}{d^k(t)}|_{t=0} = Q^k$.

Proof. $\frac{dM(t)e^{-tQ}}{dt} = 0$, $M(t)e^{-tQ}$ is constant. $M(t)$ is any matrix satisfying the forward equation. □

Theorem 2.4. *A finite matrix Q is generator matrix for a CTMC iff $P(t) = e^{tQ}$ is a stochastic matrix for all $t \geq 0$.*

Proof. $P(t) = I + tQ + O(t^2)$ $(f(t) = O(t) \Rightarrow \frac{f(t)}{t} \leq c$, for small $t,\ c < \infty$). $q_{ij} \geq 0$ if and only if $P_{ij}(t) \geq 0,\ \forall i \neq j$ and $t \geq 0$ sufficiently small. $P(t) = P(\frac{t}{n})^n$. Note that if Q has zero row sums, Q^n also has zero row sums.

$$\sum_j [Q^n]_{ij} = \sum_j \sum_k [Q^{n-1}]_{ik} Q_{kj} = \sum_j \sum_k Q_{kj} [Q^{n-1}]_{ik} = 0.$$

$$\sum_j P_{ij}(t) = 1 + \sum_{n \in \mathbb{N}} \frac{t^n}{n!} \sum_j [Q^n]_{ij} = 1 + 0 = 1.$$

Conversely $\sum_j P_{ij}(t) = 1,\ \forall t \geq 0$, then $\sum_j Q_{ij} = \frac{dP_{ij}(t)}{dt} = 0$. □

2.1 Kolmogorov Differential Equations

Lemma 2.5. *For a CTMC with transition probability matrix $P(t)$, the following properties hold.*

1. *Limiting values of transition probabilities are related to generator matrix Q, i.e.*
$$\lim_{t\downarrow 0}\frac{P(t)-I}{t}=Q.$$

2. *Transition probability matrix has the semi group property, i.e.*
$$P(t+s)=P(t)P(s),\quad s,t\geq 0.$$

2.2 Chapman Kolmogorov Equation for CTMC

Theorem 2.6 (Kolmogorov Backward equation). *For a CTMC with transition probability matrix $P(t)$, we have*
$$\frac{dP(t)}{dt}=QP(t),\quad t\geqslant 0.$$

Proof. Since, transition probability matrix $P(t)$ for a CTMC has a semi-group property, we can write
$$\frac{P(t+h)-P(t)}{h}=\frac{(P(h)-I)}{h}P(t)=P(t)\frac{(P(h)-I)}{h}.$$
Taking limits $h\downarrow 0$, we get
$$\frac{dP_{ij}(t)}{dt}=\sum_{k\neq i}q_{ik}P_{kj}(t)-\nu_i P_{ij}(t).$$
Now the exchange of limit and summation has to be justified. For any finite $k<N$, we have
$$\liminf_{h\downarrow 0}\sum_{k\neq i}\frac{P_{ik}(h)}{h}P_{kj}(t)\geq\sum_{k\neq i,k<N}\liminf_{h\downarrow 0}\frac{P_{ik}(h)}{h}P_{kj}(t)=\sum_{k\neq i,k<N}q_{jk}P_{kj}(t).$$
Since, above is true for any finite N, taking supremum over all N, we get
$$\liminf_{h\downarrow 0}\sum_{k\neq 1}\frac{P_{ik}(h)}{h}P_{kj}(t)\geq\sum_{k\neq i}q_{jk}P_{kj}(t).$$
Conversely,
$$\begin{aligned}\limsup_{h\downarrow 0}\sum_{k\neq i}\frac{P_{ik}(h)}{h}P_{kj}(t)&\leq\limsup_{h\downarrow 0}\left(\sum_{k\neq i,k<N}\frac{P_{ik}(h)}{h}P_{kj}(t)+\sum_{k\geq N}\frac{P_{ik}(h)}{h}\right)\\&=\limsup_{h\downarrow 0}\left(\sum_{k\neq i,k<N}\frac{P_{ik}(h)}{h}P_{kj}(t)+\frac{1-P_{ii}(h)}{h}-\sum_{k\neq i,k<N}\frac{P_{ik}(h)}{h}\right)\\&=\sum_{k\neq i,k<N}q_{ik}P_{kj}(t)+\nu_i-\sum_{k\neq i,k<N}q_{ik}.\end{aligned}$$
□

Theorem 2.7. ***Kolmogorov Forward Equation:*** *Under suitablle reqularity conditions, $P'_{ij}(t)=\sum_{k\neq i}P_{ik}(t)q_{kj}-P_{ij}(t)\nu_i$, i.e. $\frac{dP(t)}{dt}=P(t)Q$.*