

Lecture 15 : Limiting Probabilities and Uniformization

1 Limiting Probabilities

We denote by $\{S_n : n \in \mathbb{N}_0\}$ the jump times of CTMC and the probability transition of the embedded Markov chain is denoted by $P = \{p_{ij} : i \neq j \in I\}$.

Definition 1.1. If the embedded Markov chain is irreducible and positive recurrent then the stationary distribution α of the embedded Markov chain is a unique solution to

$$\alpha = \alpha P, \quad \sum_i \alpha_i = 1. \quad (1)$$

We denote the mean of inter-arrival time by

$$\frac{1}{\nu_i} = \mathbb{E}[S_{n+1} - S_n | X(S_n) = i].$$

Definition 1.2. First return time to a state i for a stochastic process $\{X(t), t \geq 0\}$ is

$$T_{ii} = \inf\{t > S_1 : X(0) = i, X(t) = i\}.$$

Theorem 1.3. For a CTMC with the irreducible and positive recurrent embedded Markov chain with the stationary distribution α , following holds for the limiting process,

$$\pi_j \triangleq \lim_{t \rightarrow \infty} P_{ij}(t) = \frac{\alpha_j / \nu_j}{\sum_j \alpha_j / \nu_j}, \quad (2)$$

From (9) and (1), we see that $\{\pi_j\}$ is the unique non-negative solution to

$$\nu_j \pi_j = \sum_i \nu_i \pi_i P_{ij}, \quad \sum_j \pi_j = 1. \quad (3)$$

Proof. For any CTMC on state space I , return to a state $i \in I$ is a renewal process. We can define an alternating renewal process with on and off times characterized by the time CTMC spends in state j and otherwise respectively. Therefore, by alternating delayed renewal process theory, we have

$$\lim_{t \rightarrow \infty} \Pr\{\text{CTMC is on at time } t\} = \lim_{t \rightarrow \infty} P_{ij}(t) = \frac{\mathbb{E}\tau_j}{\mathbb{E}T_{jj}}.$$

The time spent in state $j \in I$ during k^{th} visit to this state is defined by

$$\tau_j(k) = \inf\{t > 0 : X() = j\}.$$

Number of visits to state j in the first m transitions of the CTMC is defined by

$$N_j(m) = \sum_{l=1}^m 1_{\{X(S_l)=j\}}.$$

The proportion of time in state j during the first m transitions of the CTMC is

$$\beta(m) = \frac{\frac{N_j(m)}{m} \sum_{k=1}^{N_j(m)} \frac{\tau_j(k)}{N_j(m)}}{\sum_j \frac{N_j(m)}{m} \sum_{k=1}^{N_j(m)} \frac{\tau_j(k)}{N_j(m)}}.$$

Since $N_j(m) \rightarrow \infty$ as $m \rightarrow \infty$, it follows from the strong law of large numbers that

$$\lim_{m \in \mathbb{N}} \frac{\sum_{k=1}^{N_j(m)} \tau_j(k)}{N_j(m)} = \mathbb{E}\tau_j, \quad \lim_{m \in \mathbb{N}} \frac{N_j(m)}{m} = \alpha_j.$$

□

Remark 1.4. Limiting probability π_j is long-run proportion of time the process is in state j .

Lemma 1.5. *For an irreducible CTMC with transition probability $P(t)$, if the initial state is chosen according to the limiting probability distribution π , then the resultant process will be stationary. That is,*

$$\pi P(t) = \pi, \text{ for all } t > 0.$$

Proof.

$$\sum_i P_{ij}(t)\pi_i = \sum_i P_{ij}(t) \lim_{s \rightarrow \infty} P_{ki}(s) = \lim_{s \rightarrow \infty} \sum_i P_{ij}(t)P_{ki}(s) = \lim_{s \rightarrow \infty} P_{kj}(t+s) = \pi_j.$$

□

Remark 1.6. Another way of arriving at the limiting probabilities are by forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj}P_{ik}(t) - \nu_i P_{ij}(t). \quad (4)$$

Assume that the limiting probabilities exist. Then, it is easy to observe that $P'_{ij}(t) \rightarrow 0$ as $t \rightarrow \infty$. Letting $t \rightarrow \infty$, assuming that the limit and summation can be exchanged, we get the expression for π_j .

Remark 1.7. In any interval $(0, t)$, the number of transitions into state j must equal to within 1 the number of transitions out of state j . Hence, in the long run, The rate at which transitions occur into state j = The rate at which transitions occur out of state j . That is,

$$\nu_i \pi_i = \sum_i \pi_i q_{ij}. \quad (5)$$

Hence,

$$\nu_i P_{ij} = \sum_i \pi_i q_{ij}, \quad \sum_j \pi_j = 1. \quad (6)$$

are called balance equations.

2 Uniformization

Consider a continuous-time Markov chain in which the mean time spent in a state is the same for all states. That is, say $\nu_i = \nu$ for all states i . Let $N(t)$ denote the number of state transitions by time t . Since the amount of time spent in each state is exponential ν , $\{N(t), t \geq 0\}$ is a Poisson process with parameter ν . To compute the transition probabilities $P_{ij}(t)$, we can condition on $N(t)$ as follows:

$$\begin{aligned} P_{ij}(t) &= Pr(X(t) = j | X(0) = i) \\ &= \sum_{n \in \mathbb{N}_0} Pr(X(t) = j, N(t) = n | X(0) = i) \\ &= \sum_{n \in \mathbb{N}_0} Pr(N(t) = n | X(0) = i) Pr(X(t) = j | X(0) = i, N(t) = n). \end{aligned}$$

Hence,

$$P_{ij}(t) = \sum_{n \in \mathbb{N}_0} P_{ij}^{(n)} e^{-\nu t} \frac{(\nu t)^n}{n!}.$$

The above equation helps to compute $P_{ij}(t)$ approximately by computing an appropriate partial sum. But its application is limited as the rates are all assumed to be equal. But it so turns out that most Markov chains can be put in that form by allowing hypothetical transitions from a state to itself.

2.1 Uniformization step

Consider a CTMC with bounded ν_i s. Choose ν such that

$$\nu_i \leq \nu, \tag{7}$$

for all i . Since from each stage, the Markov chain leaves at rate ν_i , we could equivalently assume that the transitions occur at a rate ν but only $\frac{\nu_i}{\nu}$ are real transitions and the remaining transitions are fictitious. Any Markov chain satisfying (7) can be thought of as being in a process that spends an exponential amount of time with rate ν in state i and then makes a transition to state j with probability P_{ij}^* , where

$$P_{ij}^* = \begin{cases} 1 - \frac{\nu_i}{\nu} & : j = i \\ \frac{\nu_i}{\nu} P_{ij} & : j \neq i. \end{cases} \tag{8}$$

The transition probabilities are computed by

$$P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^{*n} e^{-\nu t} \frac{(\nu t)^n}{n!}$$

This technique of uniformizing the rate in which a transition occurs from each state to state by introducing self transitions is called uniformization.

2.2 Semi Markov Processes

Consider a stochastic process with states $0, 1, 2, \dots$ such that whenever it enters state i ,

1. The next state it enters is state j with probability P_{ij} .

2. Given the next state is going to be j , the time until the next transition from state i to state j has distribution F_{ij} . If we denote the state at time t to be $Z(t)$, $\{Z(t), t \geq 0\}$ is called a Semi Markov process.

Markov chain is a semi Markov process with

$$F_{ij}(t) = \begin{cases} 0 & : t \leq 1 \\ 1 & : t > 1. \end{cases}$$

Let H_i the distribution of time the semi Markov process stays in state i before transition. We have $H_j(t) = \sum_i P_{ij} F_{ij}(t)$, $\mu_i = \int_0^\infty x dH_i(x)$. Let X_n denote the n^{th} state visited. Then $\{X_n\}$ is a Markov chain with transition probability P called the embedded Markov chain of the semi Markov process.

Definition: If the embedded Markov chain is irreducible, then Semi Markov process is said to be irreducible. Let T_{ii} denote the time between successive transitions to state i . Let $\mu_{ii} = E[T_{ii}]$.

Theorem 2.1. *If semi Markov process is irreducible and if T_{ii} has non-lattice distribution with $\mu_{ii} < \infty$ then,*

$$\pi_i = \lim_{t \rightarrow \infty} P(Z(t) = i | Z(0) = j)$$

exists and is independent of the initial state. Furthermore, $\pi_i = \frac{\mu_i}{\mu_{ii}}$.

Corollary 4.8.2 *If the semi-Markov process is irreducible and $\mu_{ii} < \infty$, then with probability 1, $\frac{\mu_i}{\mu_{ii}} = \lim_{t \rightarrow \infty} \frac{\text{Amount of time in } [0,t]}{t}$ a.s.*

Theorem 2.2. *Suppose conditions of the previous theorem hold and the embedded Markov chain is positive recurrent. Then $\pi_i = \frac{\alpha_i \mu_i}{\sum_j \alpha_j \mu_j}$.*

Proof. Define the notation as follows:

$Y_i(j)$ = amount of time spent in state i during j^{th} visit to that state. $i, j \geq 0$.

$N_i(m)$ = number of visits to state i in the first m transitions of the semi-Markov process.

The proportion of time in i during the first m transitions:

$$\begin{aligned} P_{i=m} &= \frac{\sum_{j=1}^{N_i(m)} Y_i(j)}{\sum_l \sum_{j=1}^{N_l(m)} Y_l(j)} \\ &= \frac{\frac{N_i(m)}{m} \sum_{j=1}^{N_i(m)} Y_i(j)}{\sum_l \frac{N_l(m)}{m} \sum_{j=1}^{N_l(m)} Y_l(j)} \end{aligned}$$

Since $N_i(m) \rightarrow \infty$ as $m \rightarrow \infty$, it follows from the strong law of large numbers that $\frac{\sum_{j=1}^{N_i(m)} Y_i(j)}{N_i(m)} \rightarrow \mu_i$ and $\frac{N_i(m)}{m} \rightarrow (E[\text{number of transitions between visits to state } i])^{-1} = \alpha_i$. Letting $m \rightarrow \infty$, result follows. \square

Theorem 2.3. *If Semi Markov process is irreducible and non lattice, then $\lim_{t \rightarrow \infty} P(Z(t) = i, Y(t) > x, S(t) = j | Z(0) = k) = \frac{P_{ij} \int_x^\infty F_{ij}^c(y) d(y)}{\mu_{ii}}$. Let $Y(t)$ denote the time from t until the next transition. $S(t)$ state entered at the first transition after t .*

Proof. The trick lies in defining the "ON" time.

$$E[\text{ON time in a cycle}] = E[(X_{ij} - x)^+].$$

□

Corollary:

$$\lim_{t \rightarrow \infty} P(Z(t) = i, Y(t) > x | Z(0) = k) = \frac{\int_x^\infty H_i^c(y) d(y)}{\mu_{ii}}.$$

Since a Continuous Time Markov Chain (CTMC) is a semi-Markov chain with $F_{ij}(t) = 1 - e^{-\nu_i t}$. From the theory of semi-Markov process, if the embedded Markov chain with transition probabilities P_{ij} is irreducible and positive recurrent, then the limiting process,

$$\pi_j \triangleq \lim_{t \rightarrow \infty} P_{ij}(t) = \frac{\alpha_j / \nu_j}{\sum_j \alpha_j / \nu_j}, \quad (9)$$

where α_i is the stationary distribution of the embedded Markov chain. From 9 and 1, we see that $\{\pi_j\}$ is the unique non-negative solution to

$$\nu_j \pi_j = \sum_i \nu_i \pi_i P_{ij}, \quad \sum_j \pi_j = 1. \quad (10)$$

Remark 2.4. From the theory of semi-Markov process, it also follows that π_j also equals the long-run proportion of time the process is in state j .

Remark 2.5. If the initial state is chosen according to the limiting probabilities $\{\pi_j\}$ then the resultant process will be stationary. That is,

$$\sum_i \pi_i P_{ij}(t) = \pi_j, \quad \text{for all } t.$$

Proof.

$$\begin{aligned} \sum_i P_{ij}(t) \pi_i &= \sum_i P_{ij}(t) \lim_{s \rightarrow \infty} P_{ki}(s) \\ &= \lim_{s \rightarrow \infty} \sum_i P_{ij}(t) P_{ki}(s) \\ &= \lim_{s \rightarrow \infty} P_{ki}(t + s) \\ &= \pi_j. \end{aligned}$$

□