Lecture 16: Reversibility

1 Reversibility

Definition 1.1. A stochastic process $X(t)$ is **reversible** if $(X(t_i) : i \in [n])$ has the same distribution as $(X(\tau - t_i) : i \in [n])$ for all $t_i, \tau \in I, i \in [n]$.

Lemma 1.2. *A reversible process is stationary.*

Proof. Since $X(t)$ is reversible, both $(X(t_i) : i \in [n])$ and $(X(\tau + t_i) : i \in [n])$ have the same distribution as $(X(-t_i) : i \in [n])$. □

Theorem 1.3. *A stationary Markov chain with state space I and probability transition matrix P is reversible iff there exists a probability distribution π, that satisfy the detailed balanced conditions*

$$\pi_i P_{ij} = \pi_j P_{ji}, \quad \forall i, j \in I. \tag{1}$$

When such a distribution π exists, it is the equilibrium distribution of the process.

Proof. We assume that $X(t)$ is reversible, and hence stationary. We denote the stationary distribution by π, and by reversibility of $X(t)$ we have

$$\Pr\{X(t) = i, X(t+1) = j\} = \Pr\{X(t) = j, X(t+1) = i\},$$

and hence we obtain the detailed balanced conditions (1).

Conversely, let π be the distribution that satisfies the detailed balanced conditions, then summing up both sides over $j \in I$, we see that this distribution is the equilibrium distribution. Let $j_i \in I$ for $i \in [m]$, and we write

$$\Pr\{X(t+i-1) = j_i, i \in [m]\} = \pi(j_0) \prod_{i=1}^{m} P_{(j_{i-1}}, j_i),$$

$$\Pr\{X(t'+i-1) = j_{m-i+1}, i \in [m]\} = \pi(i_m) \prod_{i=m}^{1} P(j_i, j_{i-1}).$$

From detailed balanced equations (1) it follows that RHS of above two equations are identical. Taking $\tau = t + t' + m$, we deduce that $X(t)$ is reversible. □

Theorem 1.4. *A stationary Markov process with state space I and generator matrix Q is reversible iff there exists a probability distribution π, that satisfy the detailed balanced conditions*

$$\pi_i Q_{ij} = \pi_j Q_{ji}, \quad \forall i, j \in I. \tag{2}$$

When such a distribution π exists, it is the equilibrium distribution of the process.

Proof. We assume that $X(t)$ is reversible, and hence stationary. We denote the stationary distribution by π, and by reversibility of $X(t)$ we have

$$\Pr\{X(t) = i, X(t+\tau) = j\} = \Pr\{X(t) = j, X(t+\tau) = i\},$$

and hence we obtain the detailed balanced conditions (2) by taking limit $\tau \to 0$.

Conversely, let π be the distribution that satisfies the detailed balanced conditions, then summing up both sides over $j \in I$, we see that this distribution is the equilibrium distribution. Consider now the behavior of stationary process $X(t)$ in $[-T, T]$. Process may start at time $-T$ in state j_1 and sees m states by time T. For $i \in [m-1]$, we can define

$$S_1 = -T, \qquad S_{i+1} = \inf\{t > S_i : X(t) \neq X(S_i)\}, \qquad S_{m+1} = T.$$

That is, the process spends period $S_{i+1} - S_i$ in state j_i for $i \in [m]$, and transitions to state j_{i+1} at instant S_{i+1} for $i \in [m-1]$. Probability of this event is

$$\Pr\{X(t) = j_i, \quad t \in [S_i, S_{i+1}), i \in [m]\} = \pi(j_1) \prod_{i=1}^{m-1} Q(j_i, j_{i+1}) \prod_{i=1}^{m} e^{-\nu(j_i)(S_{i+1}-S_i)}.$$

Consider the stationary process that start in state j_m at time $\tau - T$ such that, for $i \in [m]$

$$X(t) = j_i, \;\; t \in [\tau - S_{i+1}, \tau - S_i).$$

Probability of this event is

$$\Pr\{X(t) = j_i, \quad t \in [\tau - S_{i+1}, \tau - S_i), i \in [m]\} = \pi(j_m) \prod_{i=2}^{m} Q(j_i, j_{i-1}) \prod_{i=1}^{m} e^{-\nu(j_i)(S_{i+1}-S_i)}.$$

From detailed balance equation (2) it follows that

$$\pi(j_1) \prod_{i=1}^{m-1} Q(j_i, j_{i+1}) = \pi(j_m) \prod_{i=2}^{m} Q(j_i, j_{i-1}).$$

Hence, it follows that $X(t)$ is reversible. □

Definition 1.5. **Probability flux** from state i to state j is defined as $\pi_i Q_{ij}$.

Lemma 1.6. *For a stationary Markov process, probability flux balances across a cut $A \subseteq I$, that is*

$$\sum_{i \in A} \sum_{j \notin A} \pi_i Q_{ij} = \sum_{i \in A} \sum_{j \notin A} \pi_j Q_{ji}.$$

Proof. From full balance condition $\pi Q = 0$, we get

$$\sum_{j \in A} \sum_{i \in I} \pi_i Q_{ij} = \sum_{j \in A} \sum_{i \in I} \pi_j Q_{ji} = 0.$$

Further, we have the following identity

$$\sum_{j \in A} \sum_{i \in A} \pi_i Q_{ij} = \sum_{j \in A} \sum_{i \in A} \pi_j Q_{ji}.$$

Subtracting the second identity from the first, we get the result. □

Remark 1.7. For $A = \{i\}$, the above equation reduces to full balance equations

$$\sum_{i \in I} \pi_i Q_{ij} - \pi_j Q_{jj} = \sum_{i \neq j} \pi_i Q_{ij} = \sum_{i \neq j} \pi_j Q_{ji} = -\pi_j Q_{jj}.$$

Example 1.8 (An Ergodic Random Walk). Any ergodic, positive recurrent random walk is time reversible. The transition probability matrix is $P_{i,i+1} + P_{i-1,i} = 1$. For every n transitions from $i+1$ to i, there must be at least $n-1$ transitions from i to $i+1$. The rate of transitions from $i+1$ to i must hence be same as the number of transitions from i to $i+1$. So the process is time reversible.

Proposition 1.9. *An ergodic birth and death process is time reversible in steady state.*

Proof. To prove the above, we must show that the rate at which the process goes from state i to $i+1$ is equal to the rate of going from $i+1$ to i. But during any time interval of length t, the number of transitions from i to $i+1$ should be within 1 of the number of transitions from $i+1$ to i (since the process is birth and death process. Hence, as $t \to \infty$, both rates will be equal. □

Example 1.10 (The Metropolis Algorithm). Let $\{a_j \in \mathbb{R}_+,\ j \in [m]\}$ be set of positive numbers and let $A = \sum_{i=1}^m a_i$. Suppose our main goal is to simulate a sequence of independent random variables with $\pi_j = \frac{a_j}{A}$, where m is large and A is difficult to compute directly. To generate such a sequence of random variables whose distribution converges to π, we simulate a Markov chain whose limiting probabilities are π. Let Q be an irreducible transition probability matrix on the integers $[n]$ such that $Q = Q^T$. Generate a Markov chain $\{X_n\}$ such that the transition probabilities are given by

$$P_{ij} = \begin{cases} Q_{ij} \min\left(1, \frac{a_j}{a_i}\right), & j \neq i, \\ Q_{ii} + \sum_{j \neq i} Q_{ij} \left\{1 - \min\left(1, \frac{a_j}{a_i}\right)\right\}, & j = i. \end{cases}$$

It can be directly verified that the chain is irreducible and that π is the equilibrium distribution.

Definition 1.11. Consider a finite undirected graph $G = (I, E)$ with edge weights $w : E \to \mathbb{R}_+$. We can consider a random walk on this graph with states being location of particle on one of the nodes of this graph. Probability of movement of this particle from node i to node j on edge $E = (i, j)$ is defined by

$$P_{ij} = \frac{w_{ij}}{\sum_{\{i,k\} \in E} w_{ik}}.$$

The Markov chain describing the sequence of vertices visited by the particle is called a **random walk on an edge weighted graph**.

Lemma 1.12. *Reversible Markov chain is equivalent to random walk on undirected graphs.*

Proof. First we show that random walk on undirected graphs is a reversible Markov chain. Let

$$\pi_i = \frac{w_i}{w_G}, \text{ where } w_G = \sum_{i \in I} w_i = \sum_{i \in I} \sum_{\{i,j\} \in E} w_{ij}.$$

Then, it is easy to check that this is an equilibrium distribution and

$$\pi_i P_{ij} = \pi_j P_{ji}.$$

Conversely, let $X(t)$ be a reversible Markov chain on finite state-space I and transition matrix P. We create a graph $G = (I, E)$, where $\{i, j\} \in E$ if $P_{ij} > 0$. We define

$$w_{ij} \triangleq \pi_i P_{ij} = \pi_j P_{ji} = w_{ji}.$$

With this choice of weights $w_i = \pi_i$, the transition matrix associated with this network is P. □

1.1 Necessary condition for time reversibility

If we try to prove the equations necessary for time reversibility, $x_i P_{ij} = x_j P_{ji}$ for all $i, j \in I$, for any arbitrary Markov chain, one may not end up getting any solution. This is so because, if $P_{ij} P_{jk} > 0$, then $\frac{x_i}{x_k} = \frac{P_{kj} P_{ji}}{P_{ij} P_{jk}} \neq \frac{P_{ki}}{P_{ik}}$.
Thus we see that a necessary condition for time reversibility is $P_{ij} P_{jk} P_{ki} = P_{ik} P_{kj} P_{ji},\ \forall i, j, k$. In fact we can show the following.

Theorem 1.13. *A stationary Markov chain is time reversible if and only if starting in state i, any path back to state i has the same probability as the reversed path, for all i. That is, if*

$$P_{ii_1} P_{i_1 i_2} \dots P_{i_k i} = P_{i,i_k} P_{i_k i_{k-1}} \dots P_{i_1,i}.$$

Proof. The proof of necessity is as indicated above. To see the sufficiency part, fix states i, j

$$\sum_{i_1, i_2, \dots i_k} P_{ii_1} \dots P_{i_k,j} P_{j,i} = \sum_{i_1, i_2, \dots i_k} P_{i,j} P_{j,i_k} \dots P_{i_1 i}$$

$$(P^k)_{ij} P_{ji} = P_{ij} (P^k)_{ji}$$

$$\frac{\sum_{k=1}^n (P^k)_{ij} P_{ji}}{n} = \frac{\sum_{k=1}^n P_{ij} (P^k)_{ji}}{n}$$

As limit $n \to \infty$, we get the desire result. □

2 Reversed Processes

Definition 2.1. Let $X(t)$ be a stochastic process then $X(\tau - t)$ is the reversed process.

Lemma 2.2. *If $X(t)$ is a time homogeneous non-stationary Markov chain then the reversed process $X(\tau - t)$ is a non time-homogenous Markov chain.*

Proof. Let $\mathcal{F}_m = \cup_{k \geq m} \{X_k = i_k\}$. Then, we can write

$$\Pr\{X_{m-1} = i | X_m = j, \mathcal{F}_{m+1}\} = \frac{\Pr\{X_{m-1} = i | X_m = j\} \Pr\{\mathcal{F}_{m+1} | X_{m-1} = i, X_m = j\}}{\Pr\{\mathcal{F}_{m+1} | X_m = j\}}.$$

Result follows from Markov property of $X(t)$, i.e.

$$Pr\{\mathcal{F}_{m+1} | X_m = j, X_{m-1} = i\} = \Pr\{\mathcal{F}_{m+1} = i | X_m = j\}.$$

□

Lemma 2.3. *If $X(t)$ is a stationary Markov process with generator matrix Q and equilibrium distribution π, then the reversed process $X(\tau - t)$ is a stationary Markov process with same equilibrium distribution π and generator matrix Q^* such that*

$$Q^*_{ij} = \frac{\pi_j}{\pi_i} Q_{ji}.$$

Proof. Easy to verify from definition of reversibility that

$$\Pr\{X(t+h) = j, X(t) = i\} = \Pr\{X(t+h) = i, X(t) = j\}.$$

Also, it's easy to check that $\pi Q^* = 0$. □

Lemma 2.4. *A stationary Markov process with generator matrix Q is reversible if the reverses process follows the same probabilistic law as the original process, i.e. $Q^* = Q$. Any non-negative vector π satisfying $\pi_i Q_{ij} = \pi_j Q_{ji},\ \forall i, j \in I$ and $\sum_{j \in I} \pi_j = 1$ is stationary distribution of this Markov process.*

2.1 Example 4.7(E): Example 4.3(C) revisited

Example 4.3(C) was with regard to the age of a renewal process. X_n the forward process there was such that it increases in steps of 1 until it hits a value chosen by the inter arrival distribution. Hence the reverse process should be such that it decreases in steps of 1 until it hits 1 and then jumps to a state as chosen by the inter arrival distribution. Thus letting π_i as the probability of inter arrival, it seems likely that $P_{1i}* = \pi_i,\ P_{i,i-1} = 1,\ i > 1$. We have that $P_{i,1} = \frac{\pi_i}{\sum_{j\geq 1}\pi_j} = 1 - P_{i,i+1},\ i \geq 1$. For the reversed chain to be given as above, we would need

$$\begin{aligned}
&\alpha_i P_{ij} = \alpha_j P^*_{ji} \\
&\alpha_i \frac{\pi_i}{\sum_j \pi_j} = \alpha_1 \pi_i \\
&\alpha_i = \alpha_1 P(X \geq i) \\
&1 = \sum_i \alpha_i = \alpha_1 E[X]; \alpha_i = \frac{P(X \geq i)}{E[X]},
\end{aligned}$$

where X is the inter arrival time. We need to verify that $\alpha_i P_{i,i+1} = \alpha_{i+1} P^*_{i+1,i}$ or equivalently $P(X \geq i)(1 - \frac{\pi_i}{P(X\geq i)}) = P(X \geq i)$ to complete the proof that the reversed process is the excess process and the limiting distributions are as given above. But that is immediate.

2.2 Simple Queues

Corollary 2.5. *Number of customers in a simple M/M/1 queue at equilibrium is a reversible Markov process.*

Theorem 2.6 (PASTA). *Poisson arrivals see time averages.*

Theorem 2.7 (Little's law). *Consider a stable single server queue. Let T_i be waiting time of customer i, $N(t)$ be the number of customers in the system at time t, and $A(t)$ be the number of customers that entered system in duration $[0, t)$, then*

$$\lim_{t\to\infty} \frac{\int_0^t N(u)du}{t} = \lim_{t\to\infty} \frac{\sum_{i=1}^{A(t)} T_i}{A(t)}.$$

Proof. Let $A(t), D(t)$ respectively denote the number of arrivals and departures in time $[0, t)$. Then, we have

$$\sum_{i=1}^{D(t)} T_i \leq \int_0^t N(u)du \leq \sum_{i=1}^{A(t)} T_i.$$

Further, for a stable queue we have

$$\lim_{t\to\infty} \frac{D(t)}{t} = \lim_{t\to\infty} \frac{A(t)}{t}.$$

Combining these two results, the theorem follows. □

2.3 Truncated Reversible Processes

Proposition 2.8. *A time-reversible chain with limiting probabilities $\pi_j,\ j \in I$, that is truncated to the set $A \subseteq I$ and remains irreducible is also time reversible and has limiting probabilities*

$$\pi_j^A = \frac{\pi_j}{\sum_{i\in A}\pi_i},\ j \in A.$$

Proof. We must show that

$$\pi_i^A Q_{ij} = \pi_j^A Q_{ji},\ i \in A,\ j \in A,$$

or equivalently,

$$\pi_i Q_{ij} = \pi_j Q_{ji},\ i \in A,\ j \in A.$$

But this is true as the original chain is time reversible. □

Example 2.9 (Two queues with joint waiting room). Consider two independent M/M/1 queues with arrival and service rates λ_i and μ_i respectively for $i \in [2]$. Then, joint distribution of two queues is

$$\pi(n_1, n_2) = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}, \quad n_1, n_2 \in \mathbb{N}_0.$$

Suppose both the queues are sharing a common waiting room, where if arriving customer finds R waiting customer then it leaves. In this case,

$$\pi(n_1, n_2) = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}, \quad (n_1, n_2) \in A \subseteq \mathbb{N}_0 \times \mathbb{N}_0.$$