Lecture 19 : Martingales

1 Martingales

Definition 1.1. A stochastic process $\{Z_n,\ n \in \mathbb{N}\}$ is said to be a **martingale** if

1. $\mathbb{E}[|Z_n|] < \infty, \quad \text{for all} \quad \text{n}.$
2. $\mathbb{E}[Z_{n+1}|Z_1, Z_2, \ldots Z_n] = Z_n$.

If the equality in second condition is replaced by $\leq$ or $\geq$, then the process is called **supermartingale** or **submartingale**, respectively.

Remark 1.2. Taking expectation on both sides of part 2 of the above definition, we get $\mathbb{E}[Z_{n+1}] = \mathbb{E}[Z_n]$, and hence $\mathbb{E}[Z_{n+1}] = \mathbb{E}[Z_1]$, for all n.

Example 1.3 (Simple random walk). Let $\{X_i\}$ be a sequence of independent random variables with mean 0. Let $Z_n = \sum_{i=1}^{n} X_i$. Then, $\{Z_n,\ n \in \mathbb{N}\}$ is a martingale. This is so because, $\mathbb{E}[Z_n] = 0$ and

$$\mathbb{E}[Z_{n+1}|Z_1, Z_2 \ldots Z_n] = \mathbb{E}[Z_n + X_{n+1}|Z_1, Z_2 \ldots Z_n] = Z_n.$$

Example 1.4. Let $\{X_i\}$ be a sequence of independent random variables with mean 1. Let $Z_n = \Pi_{i=1}^{n} X_i$. Then, $\{Z_n,\ n \in \mathbb{N}\}$ is a martingale. This is so because, $\mathbb{E}[Z_n] = 1$ and

$$\mathbb{E}[Z_{n+1}|Z_1, Z_2 \ldots Z_n] = \mathbb{E}[Z_n X_{n+1}|Z_1, Z_2 \ldots Z_n] = Z_n.$$

Example 1.5 (Branching Process). Let $\{X_n\}$ be a branching process. Let $X_0 = 1$. Then,

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i,$$

where Z_i represents the number of offspring of the i^{th} individual of the $(n-1)^{\text{st}}$ generation. conditioning on X_{n-1} yields, $\mathbb{E}[X_n] = \mu^n$ where μ is the mean number of offspring per individual. Then $\{Y_n = X_n/\mu^n : n \in \mathbb{N}\}$ is a martingale because $\mathbb{E}[Y_n] = 1$ and

$$\mathbb{E}[Y_{n+1}|Y_1, \ldots Y_n] = \frac{1}{\mu^{n+1}} \mathbb{E}[\sum_{i=1}^{X_n} Z_i|Y_1, \ldots Y_n] = \frac{X_n}{\mu^n} = Y_n.$$

Example 1.6 (Doob's Martingale). Let $X, Y_1, Y_2 \ldots$ be arbitrary random variables such that $\mathbb{E}[|X|] < \infty$. Then

$$Z_n = \mathbb{E}[X|Y_1, Y_2, \ldots Y_n]$$

is a martingale. The integrability condition can be directly verified, and

$$\mathbb{E}[Z_{n+1}|Y_1, Y_2, \ldots Y_n] = \mathbb{E}[\mathbb{E}[X|Y_1, \ldots Y_{n+1}]|Y_1, \ldots Y_n] = \mathbb{E}[X|Y_1, \ldots Y_n]] = Z_n.$$

Example 1.7. For any sequence of random variables $X_1, X_2 \ldots$, the random variables $X_i - \mathbb{E}[X_i|X_1 \ldots X_{i-1}]$ have zero mean. Define

$$Z_n = \sum_{i=1}^{n} X_i - \mathbb{E}[X_i|X_1, X_2, \ldots X_{i-1}]$$

is a martingale provided $\mathbb{E}[|Z_n|] < \infty$. To verify the same,

$$\begin{aligned}\mathbb{E}[Z_{n+1}|Z_1 \ldots Z_n] &= \mathbb{E}[Z_n + X_n - \mathbb{E}[X_n|X_1 \ldots X_{n-1}]] \\ &= Z_n + \mathbb{E}[X_n - \mathbb{E}[X_n|X_1 \ldots X_{n-1}]] = Z_n.\end{aligned}$$

1.1 Stopping Times

Definition 1.8. The positive integer values, possibly infinite, random variable N is said to be a **random time** for the process $\{Z_n\}$ if the event $\{N = n\}$ is determined by the random variables $Z_1 \ldots Z_n$. If $\Pr\{N < \infty\} = 1$, then the random time N is said to be a **stopping time**.

Definition 1.9. A **predictable sequence** $\{H_n : n \in \mathbb{N}\}$ for process $\{X_n\}$ is the one where H_n is completely determined by $X_1, X_2, ..., X_{n-1}$

$$\sum_{m=1}^{n} H_m(X_m - Xm - 1 = (H \cdot X)_n$$

Theorem 1.10. *Let $\{X_n, n \geq 0\}$ be a super martingale if $\{H_n \geq 0 : n \in \mathbb{N}\}$ is predictable and each H_n is bounded then $(H \cdot X)_n$ is super martingale*

Proof.

$$\begin{aligned}\mathbb{E}[(H \cdot X)_{n+1}|(H \cdot X)_1 ...(H \cdot X)_n] =&\mathbb{E}[H_{n+1}(X_{n+1} - X_n) + (H \cdot X)_n|(H \cdot X)_1 ...(H \cdot X)_n] \\ =&H_{n+1}(\mathbb{E}[X_{n+1}|(H \cdot X)_1 ...(H \cdot X)_n] - X_n) + (H \cdot X)_n \\ \leq&(H \cdot X)_n\end{aligned}$$

□

Definition 1.11. Let T be a random time for the process $\{X_n : n \in \mathbb{N}\}$, then **stopping process** $\{X_{T\wedge n}\}$ is defined as

$$X_{T\wedge n} = X_n 1_{\{n\leq T\}} + X_T 1_{\{n>T\}}.$$

Proposition 1.12. *If T is a random time for the martingale $\{X_n : n \in \mathbb{N}\}$, then the stopping process $\{X_{T\wedge n}\}$ is a martingale.*

Proof.

$$\begin{aligned}X_{T\wedge n} &= (H \cdot X)_n \;\; when \;\; H_n = 1_{\{n\leq T\}} \\ X_{T\wedge n} &= X_{T\wedge n-1} + 1_{\{n\leq T\}}(X_n - X_{n-1})\end{aligned}$$

$n \leq T$: $\quad X_{T\wedge n} = X_n$
$n > T$: Since $n > T$ gives $n-1 \geq T$ therefore $T \wedge n-1 \geq T$ which implies $X_{T\wedge n} = X_T$

$$X_{T\wedge n} = X_0 + \sum_{m=1}^{n} 1_{\{m\leq T\}}(X_m - X_{m-1})$$

It is suffice to show $\{1_{n\leq T}\}$ is a predictable sequence which is true since

$$\{n \leq T\} = \{T > n-1\} = \{T < n-1\}^c$$

Therefore from the previous theorem we have

$$\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_{T\wedge 1}] = \mathbb{E}[X_1]$$

□

Remark 1.13. For any martingale $\{X_n : n \in \mathbb{N}\}$, we have $\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_1]$, for all n. Now assume that T is a stopping time. It is immediate that

$$\Pr\left\{\lim_{n\in\mathbb{N}} X_{T\wedge n} = X_N\right\} = 1.$$

But is it true that

$$\lim_{n\in\mathbb{N}} \mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_N]?$$

It so turns out that the above is true under some additional regularity constraints only.

Theorem 1.14 (Martingale Stopping Theorem). *If T is a stopping time for a martingale $\{X_n : n \in \mathbb{N}\}$ such that either of the following conditions is true:*

(i) T is bounded,

(ii) $X_{T\wedge n}$ is uniformly bounded,

(iii) $\mathbb{E}[T] < \infty$, and for some real positive K, we have $\sup_{n\in\mathbb{N}} \mathbb{E}[|X_{n+1} - X_n||X_1 \dots X_n] < K$,

then X_T is integrable and $\lim_{n\in\mathbb{N}} \mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_T] = \mathbb{E}[X_1]$.

Proof. We show this is true for all three cases.

(i) Let K be the bound on T then for all $n \geq K$, we have $X_{T\wedge n} = X_T$, and hence it follows that

$$\mathbb{E}X_1 = \mathbb{E}X_{T\wedge n} = \mathbb{E}X_T, \ \forall n \geq K.$$

(ii) Dominated convergence theorem implies the result.

(iii) Since T is integrable and

$$X_{T\wedge n} \leq |X_1| + KT,$$

we observe that $X_{T\wedge n}$ is bounded by an integrable random variable, and hence result follows from dominated convergence theorem.

□

Corollary 1.15 (Wald's Equation). *If T is a stopping time for $\{X_i,\ i \in \mathbb{N}\}$ iid with $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[T] < \infty$, then*

$$\mathbb{E}[\sum_{i=1}^{T} X_i] = \mathbb{E}[T]\mathbb{E}[X].$$

Proof. Let $\mu = \mathbb{E}[X]$. Then $\{Z_n = \sum_{i=1}^{n}(X_i - \mu) : n \in \mathbb{N}\}$ is a martingale and hence from the Martingale stopping theorem, we have $\mathbb{E}[Z_T] = \mathbb{E}[Z_1] = 0$. But

$$\mathbb{E}[Z_T] = \mathbb{E}\sum_{i=1}^{N} X_i - \mu\mathbb{E}N.$$

Observe that condition 3 for Martingale stopping theorem to hold can be directly verified. Hence the result follows. □

Before we state and prove martingale convergence theorem, we state some results which will be used in the proof of the theorem.

Lemma 1.16. *If $\{Z_i, i \in \mathbb{N}\}$ is a submartingale and T is a stopping time such that $\Pr\{T \leq n\} = 1$ then*

$$\mathbb{E}Z_1 \leq \mathbb{E}Z_T \leq \mathbb{E}Z_n.$$

Proof. It follows from Theorem 1.14 that since T is bounded, $\mathbb{E}[Z_T] \geq \mathbb{E}[Z_1]$. Now, since T is a stopping time, we see that for $\{T = k\}$

$$\mathbb{E}[Z_n|Z_1, \ldots, Z_T, T = k] \quad = \mathbb{E}[Z_n|Z_1 \ldots Z_k, T = k] = \mathbb{E}[Z_n|Z_1 \ldots Z_k] \geq Z_k = Z_T.$$

Result follows by taking expectation on both sides. □

Lemma 1.17. *If $\{Z_n, n \in \mathbb{N}\}$ is a martingale and f is a convex function, then $\{f(Z_n), n \in \mathbb{N}\}$ is a submartigale.*

Proof. The result is a direct consequence of Jensen's inequality.

$$\mathbb{E}[f(Z_n)|Z_1, \ldots Z_n] \geq f(\mathbb{E}[Z_{n+1}|Z_1, \ldots Z_n]) = f(Z_n).$$

□

Construction 1.18. Let $X_n, n \geq 0$ be a sub martingale. Let $a < b$ and $N_0 = -1$.

$$\begin{aligned} N_{2k-1} &= \inf\{m > N_{2k-2} : X_m \leq a\}. \\ N_{2k} &= \inf\{m > N_{2k-1} : X_m \leq b\}. \end{aligned}$$

The above quantities N_{2k-1},N_{2k} are stopping times and the set containing values of m in the transition from a to b can be defined as

$$\begin{aligned} \{N_{2k-1} < m \leq N_{2k}\} &= \quad \{N_{2k-1} < m - 1\} \cap \{m > N_{2k}\}^c \\ &= \{m - 1 \geq N_{2k-1}\} \cap \{m - 1 \geq N_{2k}\}^c \end{aligned}$$

Since the above set depends on $\{m-1\}$ values instead of $\{m\}$ values, So

$$
\begin{aligned}
H_m =& 1_{\{N_{2k-1}<m\leq N_{2k}\}} \\
U_n =& \sup\{k : N_{2k} \leq n\}
\end{aligned}
$$

H_m defines a predictable sequence and U_n is the number of up crossings completed in time n.

Lemma 1.19 (Upcrossing inequality). *If $\{X_m : m \geq 0\}$ is a sub martingale,then*

$$
\begin{aligned}
(b-a)\mathbb{E}U_n \leq& \mathbb{E}Y_n - \mathbb{E}Y_0 \\
where\ \ Y_n :=& a + (X_n - a)^+
\end{aligned}
$$

Proof. Since X_n is a submartingale so is Y_n, as it is a convex function of X_n.Since each up crossing has a gain slightly more than $b-a$ the following inequality exists

$$
\begin{aligned}
(b-a)U_n \leq& (H \cdot Y)_n \\
=& \sum_{m=1}^{n} 1_{\{N_{2k-1}<m\leq N_{2k}\}}(Y_{m+1} - Ym) \\
=& \sum_{k=1}^{U_n}(Y_{N_{2k+1}} - Y_{N_{2k+1}})
\end{aligned}
$$

Now let $K_m = 1 - H_m$ then clearly,

$$Y_n - Y_0 = (H \cdot Y)_n + (K \cdot Y)_n$$

we have by the submartingale property of Y

$$\mathbb{E}[(K \cdot Y)_n] \geq \mathbb{E}[(K \cdot Y)_0] = 0$$

Therefore,

$$
\begin{aligned}
\mathbb{E}[(H \cdot Y)_n] + \mathbb{E}[(K \cdot Y)_n] =& \mathbb{E}[Y_n - Y_0] \\
\mathbb{E}[(H \cdot Y)_n] \leq& \mathbb{E}[Y_n - Y_0] \\
(b-a)\mathbb{E}U_n \leq& \mathbb{E}(Y_n - Y_0)
\end{aligned}
$$

□

Theorem 1.20 (Martingale Convergence Theorem). *If X_n is a submartingale with $\sup_{a\in A}\mathbb{E}[X_n^+] \leq \infty$ then $\lim_{n\in\mathbb{N}} X_n = X$ a.s with $\mathbb{E}[X] < \infty$.*

Proof.

$$
\begin{aligned}
(X-a)^+ \leq& X^+ + |a| \\
\mathbb{E}[U_n] \leq& \frac{\mathbb{E}[X_n^+] + |a|}{b-a}
\end{aligned}
$$

$\underset{n\to\infty}{lim} U_n = U$ since, $\mathbb{E}[X_n^+] < \infty$ gives $U < \infty$ a.s. This conclusion leads to

$$Pr\{\underset{a,b\in\mathbb{Q}}{\cup}\{\underset{n\in\mathbb{N}}{liminf} X_n < a < b < \underset{n\in\mathbb{N}}{limsup} X_n\}\} = 0;$$

From the above probability we have a.s

$$\underset{n\in\mathbb{N}}{limsup} X_n = \underset{n\in\mathbb{N}}{liminf} X_n$$

Now the Fatou's lemma in measure theory guarantees

$$\mathbb{E}[X^+] \leq \lim_{n\in\mathbb{N}} inf \mathbb{E}[X_n^+] < \infty$$

which implies $X < \infty$ almost sure. To see $X > -\infty$, we observe that

$$\begin{aligned}\mathbb{E}[X_n^-] &= \mathbb{E}[X_n^+] - \mathbb{E}[X_n] \\ &\leq \mathbb{E}[X_n^+] - \mathbb{E}[X_0]\end{aligned}$$

The above inequality comes from the submartingale property of X_n. Now from another application of Fatou's lemma gives,

$$\mathbb{E}[X^-] \leq \lim_{n\in\mathbb{N}} inf \mathbb{E}[X_n^-] \leq \sup_{n\in\mathbb{N}} \mathbb{E}[X_n^+] - \mathbb{E}[X_0] < \infty.$$

□