Lecture 20: Polya's Urn Scheme

The gambling interpretation of the stochastic integral suggests that it is natural to let the amount bet at time n depend on the outcomes of the first $n-1$ flips but not on the flip we are betting on, or on later flips. The next result shows that we cannot make money by gamblingon a fair game.

Theorem 0.1. *Let X_n be a martingale. If H_n is predictable and each H_n is bounded, then $(H \cdot X)_n$ is a martingale.*

Proof. It is easy to check that $(H \cdot X)_n \in \mathcal{F}_n$. Thhe boundedness of the H_n implies $E|(H \cdot X)_n| < \infty$ for each n. With this established, we can compute conditional expectations to conclude

$$\begin{aligned} E((H \cdot X)_{n+1}|\mathcal{F}_n) &= (H \cdot X)_n + E(H_{n+1}(X_{n+1} - X_n)|\mathcal{F}_n) \\ &= (HX)_n + H_{n+1}E(X_{n+1} - X_n|\mathcal{F}_n) \\ &= (H \cdot X)_n. \end{aligned}$$

□

since $H_{n+1} \in \mathcal{F}_n$ and $E(X_{n+1} - X_n|\mathcal{F}_n) = 0$

The last theorem can be interpreted as: you can't make money by gambling on a fair game. This conclusion does not hold if we only assume that H_n is optional, that is $H_n \in (F)_n$, since then we can base our bet on the outcome of the coin we are betting on.

Theorem 0.2. *Suppose $M_0, M_1, \dots$ is a martingale with respect to$\{\mathcal{F}_n\}$ and suppose T is a stopping time. Suppose that T is bounded, $T \leq K$. Then*

$$\mathbb{E}(M_T|\mathcal{F}_0) = M_0.$$

In particular, $\mathbb{E}(M_T) = \mathbb{E}(M_0)$.

To prove this fact, we first note that the event $\{T > n\}$ ismeasurable with respect to $\mathcal{F}_n$ (since we need only the information up through time n to determine if we have stopped by time n). Since M_T is the random variable which equals M_j if $T = j$ we can write

$$M_T = \sum_{j=0}^{K} M_j I\{T = j\}.$$

Let us take the conditional expectation with respect to $\mathcal{F}_{K-1}$,

$$\mathbb{E}(M_T|(F)_{K-1}) = \mathbb{E}(M_K I\{T = K\}|\mathcal{F}_{K-1}) + \sum_{j=0}^{K-1} \mathbb{E}(M_j I\{T = j\}|\mathcal{F}_{K-1}).$$

For $j \leq K-1, M_j I\{T = j\}$ is $\mathcal{F}_{K-1}$- measurable; hence

$$\mathbb{E}(M_j I\{T = j\}|\mathcal{F}_{K-1}) = M_j I\{T = j\}.$$

Since T is known to be no more than K, the event $\{T = K\}$ is the same as the event $\{T > K-1\}$. The latter event is measurable with respect to $\mathcal{F}_{K-1}$. Hence using equality

$$\mathbb{E}(YZ|\mathcal{F}_n) = Z\mathbb{E}(Y|\mathcal{F}_n).$$

Where Y is any random variable and Z is a random variable that is measurable with respect to finite number of random variables $X_1, X_2, \ldots, X_n$.

$$\begin{aligned}
\mathbb{E}(M_K I\{T = K\}|\mathcal{F}_{K-1}) &= \mathbb{E}(M_K I\{T > K-1\}|\mathcal{F}_{K-1}) \\
&= I\{T > K-1\}\mathbb{E}(M_K|\mathcal{F}_{K-1}) \\
&= I\{T > K-1\}M_{K-1}.
\end{aligned}$$

The last equality follows from the fact the M_n is a martingale. Therefore,

$$\begin{aligned}
\mathbb{E}(M_T|\mathcal{F}_{K-1}) &= I\{T > K-1\}M_{K-1} + \sum_{j=0}^{K-1} M_j I\{T = j\} \\
&= I\{T > K-2\}M_{K-1} + \sum_{j=0}^{K-2} M_j I\{T = j\}.
\end{aligned}$$

If we work through this argument again, this time conditioning with respect to $\mathcal{F}_{K-2}$, we gat

$$\begin{aligned}
\mathbb{E}(M_T|\mathcal{F}_{K-2}) &= \mathbb{E}(\mathbb{E}(M_T|\mathcal{F}_{K-1})|\mathcal{F}_{K-2}) \\
&= I\{T > K-3\}M_{K-2} + \sum_{j=0}^{K-3} M_j I\{T = j\}.
\end{aligned}$$

We can continue this process untill we get

$$\mathbb{E}(M_T|\mathcal{F}_0) = M_0.$$

There are many examples of interest where the stopping time T is not bounded. Suppose T is a stopping time with $\mathbb{P}\{T < \infty\} = 1$, i.e., a rule that guarantees that one stops eventually. (Note that the time associated to the martingale betting strategy satisfies this condition.) When can we conclude that $\mathbb{E}(M_T) = \mathbb{E}(M_0)$? To investigate this consider the stopping times $T_n = min\{T, n\}$. Note that

$$M_T = M_{T_n} + M_T I\{T > n\} - M_n I\{T > n\}.$$

Hence,

$$\mathbb{E}(M_T) = \mathbb{E}(M_{T_n}) + \mathbb{E}(M_T I\{T > n\}) - \mathbb{E}(M_n I\{T > n\}).$$

Since T_n is a bounded stopping time, it follows from the above that $\mathbb{E}(M_{T_n}) = \mathbb{E}(M_0)$. We would like to be able to say that the other termsdo not contribute as $n \to \infty$. The second term is not much of a problem. Since the probability of the event $\{T > n\}$ goes to 0 as $n \to \infty$, we are taking the expectation of the random variable M_T restricted to a smaller and smaller set. One can show that if $\mathbb{E}(|M_T|) < \infty$ then $\mathbb{E}(|M_T| I\{T > n\}) \to 0$.

The third term, if M_n and T are given satisfying

$$lim_{n\to\infty}\mathbb{E}(|M_n| I\{T > n\}) = 0,$$

then we will be able to conclude that $\mathbb{E}(M_T) = \mathbb{E}(M_0)$. We summarize this as follows.

Optional Sampling Theorem. Suppose $M_0, M_1, \ldots$ is a martingale with respect to $\{\mathcal{F}_n\}$ and T is a stopping time satisying $\mathbb{P}\{T < \infty\} = 1$,

$$\mathbb{E}(|M_T|) < \infty,$$

and

$$\lim_{n\to\infty} \mathbb{E}(|M_n| I\{T > n\}) = 0.$$

Then, $\mathbb{E}(M_T) = \mathbb{E}(M_0)$.

1 Polya's Urn Scheme

Suppose an urn initially contains b_0 black balls and w_0 white balls. Suppose balls are sampled from the urn one at a time, but after each draw 1 balls of the same color are returned to the urn. If first draw is a black, then replace b_0 with $b_0 + 1$ balls in the urn and w_0 with $w_0 + 1$ for white balls. The number of black balls in the first n draws would then have a $Bin(n, \frac{b_0}{b_0+w_0})$. Let B_n be the number of black balls in urn after n draws and $B_0 = b_0$. Now probability of getting black ball in first draw is

$$\mathbb{P}(B_1 = b_0 + 1) = \frac{b_0}{b_0 + w_0},$$

and probability of getting white balls in first draw is

$$\mathbb{P}(B_1 = b_0) = \frac{w_0}{b_0 + w_0}.$$

Similarly after two draws,

$$\begin{aligned}
\mathbb{P}(B_2 = b_0) &= \frac{w_0}{b_0 + w_0} \cdot \frac{w_0 + 1}{b_0 + w_0 + 1} \\
\mathbb{P}(B_2 = b_0 + 1) &= \frac{b_0}{b_0 + w_0} \cdot \frac{w_0}{b_0 + w_0 + 1} + \frac{w_0}{b_0 + w_0} \cdot \frac{b_0}{b_0 + w_0 + 1} \\
\mathbb{P}(B_2 = b_0) &= \frac{b_0}{b_0 + w_0} \cdot \frac{b_0 + 1}{b_0 + w_0 + 1}.
\end{aligned}$$

For first three draws,

$$\begin{aligned}
\mathbb{P}(first\ 3\ draws\ are\ bwb) &= \frac{b_0}{b_0 + w_0} \cdot \frac{w_0}{b_0 + w_0 + 1} \cdot \frac{b_0 + 1}{b_0 + w_0 + 2} \\
\mathbb{P}(first\ 3\ draws\ are\ bbw) &= \frac{b_0}{b_0 + w_0} \cdot \frac{b_0 + 1}{b_0 + w_0 + 1} \cdot \frac{w_0}{b_0 + w_0 + 2} \\
\mathbb{P}(first\ 3\ draws\ are\ wbb) &= \frac{w_0}{b_0 + w_0} \cdot \frac{b_0}{b_0 + w_0 + 1} \cdot \frac{b_0 + 1}{b_0 + w_0 + 2}.
\end{aligned}$$

Here b stands for black ball and w stands for white ball. I we observe above 3 equations, all of them are same. like wise

$$\begin{aligned}
\mathbb{P}(bbwww) &= \frac{b_0(b_0 + 1)w_0(w_0 + 1)(w_0 + 2)}{\prod_{i=0}^{4}(b_0 + w_0 + i)} \\
\mathbb{P}(bwwwb) &= \frac{b_0 w_0(w_0 + 1)(w_0 + 2)(b_0 + 1)}{\prod_{i=0}^{4}(b_0 + w_0 + i)}.
\end{aligned}$$

Again above two probabilities are equal.

Definition 1.1. An infinite sequence $\{X_i\}_{i=1}^{\infty}$ of random variables is exchangeable if $\forall\ n = 1, 2, \ldots$

$$X_1, \ldots, X_n = X_{\pi(1)}, \ldots, X_{\pi(n)}, \forall \pi \in S(n),$$

where $S(n)$ is the symmetric group, the group of permutations.

Polya's Urn Model is one of the examples for exchangeability. An Example would be following

$$\mathbb{P}(bbwww) = \mathbb{P}(bwwwb)$$

If $\xi_1, \xi_2, \ldots, \xi_n$ denote the sigma algebra for the color of the drawn ball i.e., ξ_i represents the color of the i^{th} ball, from the definition of excangeability

$$(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = (\xi_2, \xi_1, \xi_5, \xi_4, \xi_3).$$

Note 1.2. Polya's Urn scheme generate exchangeable sequences.

Let

$$X_n = \frac{B_n}{B_n + W_n} = \frac{B_n}{b_0 + w_0 + n}, \; 0 \leq X_n \leq 1,$$

represents the proportion of black balls after n draws, then given the past $\xi_1, \xi_2, \ldots, \xi_n$

$$B_{n+1} = \begin{cases} B_n & w.p \; (1 - \frac{B_n}{b_0+w_0+n}) \\ B_{n+1} & if \; \xi_{n+1} \; w.p \; \frac{B_n}{b_0+w_0+n}. \end{cases}$$

Now

$$\begin{aligned} \mathbb{E}[X_{n+1}|\xi_1, \xi_2, \ldots \xi_n] &= \frac{1}{b_0 + w_0 + n + 1}\mathbb{E}[B_{n+1}|\xi_1, \xi_2, \ldots \xi_n] \\ &= \frac{1}{b_0 + w_0 + n + 1}\mathbb{E}[B_n(1 - X_n) + (B_n + 1)X_n] \\ &= \frac{B_n + X_n}{b_0 + w_0 + n + 1} \\ &= X_n. \end{aligned}$$

That means it is a martingale.

Note 1.3. X_n is a martingale.

1.1 Analysis of the Polya urn model

Theorem 1.4. *(De Finetti 1931) A binary sequence $\{X_n\}_{i=1}^{\infty}$ is exchageable iff there exixtes a distribution function $F(p)$ on $[0, 1]$ such that for any $n \geq 1$,*

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \int_0^1 p^{S_n}(1 - p)^{n-S_n} dF(p)$$

where $S_n = \sum_i x_i$.

The distribution F is a function of the limiting frequency

$$Y = \bar{X}_\infty = \lim_{n\to\infty} \frac{\sum_i X_i}{n}, \mathbb{P}(Y \le p) = F(p),$$

and conditioning on $Y = p$ results in iid Bernoulli draws

$$\mathbb{P}(X_1 = x_1, \dots, X_N = x_n | Y = p) = p^{S_n}(1-p)^{n-S_n},$$

and for the Polya urn model

$$\lim_{n\to\infty} \bar{X}_n = Y \ Beta\left(\frac{B_0}{B_0+W_0}, \frac{W_0}{B_0+W_0}\right)$$

The result can be interpreted from a statistical, probabilistic and function analytic perspective.

We will use De Finetti's theorem to compute the limiting distribution for the Polya urn model

$$\lim_{n\to\infty} \bar{X}_n = Y \ Beta\left(\frac{B_0}{B_0+W_0}, \frac{W_0}{B_0+W_0}\right).$$

We first define the Beta and Gamma functions

$$\beta(x,y) = \int_0^1 p^{x-1}(1-p)^{y-1}dp, \Gamma(x+1) = x\Gamma(x), \beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

The probability of observing k black balls given n draws

$$\mathbb{P}(k \ black \ balls \ given \ n \ draws) = \binom{n}{k} \frac{B_0(B_0+1)\dots(B_0+k-1)W_0(W_0+1)\dots(W_0+n-k-1)}{(B_0+W_0)(B_0+W_0+1)\dots(B_0+W_0+n-1)} \quad (1)$$

$$= \binom{n}{k} \frac{\beta(B_0+k, B_0+n-k)}{\beta(B_0, W_0)}. \quad (2)$$

Note that the proportion of black balls at any stage n of the process as

$$\rho_n = \frac{B_n}{B_n+W_n}, \rho_\infty = \lim_{n\to\infty} \frac{B_n}{B_n+W_n}.$$

We know that

$$\mathbb{P}(k \ black \ balls \ given \ n \ draws | \rho_\infty = p) = \binom{n}{k} p^k (1-p)^{n-k},$$

and if $\rho_\infty\ F(p)$ then,

$$\mathbb{P}(k\ black\ balls\ given\ n\ draws) = \int_0^1 \mathbb{P}(k\ black\ balls\ given\ n\ draws|\rho_\infty = p)dF(p),$$

$$\mathbb{P}(k\ black\ balls\ given\ n\ draws) = \binom{n}{k}\int_0^1 p^k(1-p)^{n-k}dF(p). \tag{3}$$

By equating (2) and (3) we obtain,

$$\begin{aligned}
\int_0^1 p^k(1-p)^{n-k}dF(p) &= \frac{\beta(B_0+k, B_0+n-k)}{\beta(B_0, W_0)} \\
&= \frac{1}{\beta(B_0, W_0)}\int_0^1 p^{B_0+k-1}(1-p)^{B_0+n-k-1}dp \\
&= \int_0^1 p^k(1-p)^{n-k}\frac{p^{B_0-1}(1-p)^{W_0-1}}{\beta(B_0, W_0)}dp,
\end{aligned}$$

which gives the limiting distribution for Polya's urn scheme as

$$f(p) = \frac{1}{\beta(B_0, W_0)}p^{B_0-1}(1-p)^{W_0-1}.$$