Lecture 21: Exchangeability

1 Random Walk

Definition 1.1. Let $\{X_i : i \in \mathbb{N}\}$ be *iid* random variables with finite $\mathbb{E}[|X_1|]$. Let

$$S_n = \sum_{k=1}^{n} X_i, \quad n \in \mathbb{N}_0.$$

Then the process $\{S_n : n \in \mathbb{N}_0\}$ is called a **random walk**.

Definition 1.2. A random walk is called a **simple random walk** if

$$\Pr\{X_1 = 1\} = 1 - \Pr\{X_1 = -1\}.$$

Remark 1.3. A simple random walk has the interpretation of the winnings of a gambler who plays a simple coin toss game and wins Rupee 1 if heads and loses Rupee 1 if tails.

Remark 1.4. Random walks are useful in analyzing GI/GI/1 Queues, Ruin systems and even stock prices.

Definition 1.5. Let X_i belong to probability space $(S, \mathcal{S}, \mu)$. Consider the probability space $(\Omega, \mathcal{F}, P)$ for process $\{X_i : i \in \mathbb{N}\}$ where

$$\Omega = \prod_{i \in \mathbb{N}} S, \qquad \mathcal{F} = \prod_{i \in \mathbb{N}} \mathcal{S}, \qquad P = \prod_{i \in \mathbb{N}} \mu.$$

A **finite permutation** of $\mathbb{N}$ is a map $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(i) \neq i$ for only finitely many i.

Definition 1.6. For a finite permutation π, we define $(\pi\omega)_i = \omega_{\pi(i)}$ for all $i \in \mathbb{N}$.

Definition 1.7. An event A is **permutable** if $A = \pi^{-1}A = \{\omega \in \Omega : \pi\omega \in A\}$ for any finite permutation π.

Definition 1.8. The collection of permutable events is a σ-field called the **exchangeable** σ-field and denoted by $\mathcal{E}$.

Definition 1.9. A sequence X of random variables is called **exchangeable** if for each n and permutation $\pi : [n] \to [n]$, joint distribution of $(X_1, X_2, \dots, X_n)$ and $(X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(n)})$ are same.

Example 1.10. Suppose balls are selected randomly, without replacement, from an urn consisting of n balls of which k are white. For $i \in [n]$, let

$$X_i = 1_{\{i^{\text{th}} \text{ selection is white}\}},$$

then $(X_1, \dots X_n)$ will be exchangeable but not independent. In particular, let $A = \{i \in [n] : X_i = 1\}$. Then, we know that $|A| = k$, and we can write

$$\Pr\{X_i = 1, i \in A, X_j = 0, j \in A^c\} = \Pr\{A = (i_1, i_2, \dots, i_k)\} = \frac{(n-k)!k!}{n!} = \frac{1}{\binom{n}{k}}.$$

This joint distribution is independent of set of exact locations A, and hence exchangeable. Further, we can show that all X_i are identically distributed, since

$$\Pr\{X_1 = 1, X_2, \dots, X_n\} = \Pr\{X_i = 1, X_1, \dots, X_{i-1}, X_i, \dots, X_n\}.$$

Further, it can be seen that

$$\Pr\{X_2 = 1 | X_1 = 1) = \frac{k-1}{n-1} \neq \frac{k}{n-1} = \Pr\{X_2 = 1 | X_1 = 0\}.$$

Example 1.11. Let Λ denote a random variable having distribution G. Let X be a sequence of dependent random variables, where each of these random variables are conditionally *iid* with distribution F_λ given $\Lambda = \lambda$.Then, these random variables are exchangeable since

$$\Pr\{X_1 \leq x_1 \dots, X_n \leq x_n\} = \int_\lambda \prod_{i=1}^n F_\lambda(x_i) dG(\lambda),$$

which is symmetric in $(x_1, \dots x_n)$.

Theorem 1.12 (de Finetti's Theorem). *If X is an exchangeable sequence of random variables then conditioned on $\mathcal{E}$, each random variable X_i in the sequence is iid.*

Proof. Let $I_{n,k} = \{i \subseteq [n]^k : i_j \text{ distinct}\}$. Then, for a function $\phi : \mathbb{R}^k \to \mathbb{R}$, we can define

$$A_n(\phi) = \frac{1}{(n)_k} \sum_{i \in I_{n,k}} \phi(X_{i_1}, X_{i_2}, \dots, X_{i_k}),$$

where $(n)_k = n(n-1)\dots(n-k+1)$. Clearly, $A_n(\phi) \in \mathcal{E}_n$ measurable and hence, $\mathbb{E}[A_n(\phi)|\mathcal{E}_n] = A_n(\phi)$. Since, X is exchangeable, we have

$$A_n(\phi) = \frac{1}{(n)_k} \sum_{i \in I_{n,k}} \mathbb{E}[\phi(X_{i_1}, X_{i_2}, \dots, X_{i_k})|\mathcal{E}_n] = \mathbb{E}[\phi(X_1, X_2, \dots, X_k)|\mathcal{E}_n].$$

Since $\mathcal{E}_n \to \mathcal{E}$, we have

$$\lim_{n\in\mathbb{N}} A_n(\phi) = \lim_{n\in\mathbb{N}} \mathbb{E}[\phi(X_1, X_2, \dots, X_k)|\mathcal{E}_n] = \mathbb{E}[\phi(X_1, X_2, \dots, X_k)|\mathcal{E}].$$

Let f and g be bounded functions on $\mathbb{R}^{k-1}$ and $\mathbb{R}$ respectively, such that $\phi(x_1, \dots, x_k) = f(x_1, \dots, x_{k-1})g(x_k)$. We also define $\phi_j(x_1, \dots, x_{k-1}) = f(x_1, \dots, x_{k-1})g(x_j)$, to write

$$\begin{aligned}
(n)_{k-1} A_n(f) n A_n(g) &= \sum_{i \in I_{n,k-1}} f(X_{i_1}, \dots, X_{i_{k-1}}) \sum_m g(X_m) \\
&= \sum_{i \in I_{n,k}} f(X_{i_1}, \dots, X_{i_{k-1}}) g(X_{i_k}) + \sum_{i \in I_{n,k-1}} \sum_{j=1}^{k-1} f(X_{i_1}, \dots, X_{i_{k-1}}) g(X_{i_j}) \\
&= (n)_k A_n(\phi) + \sum_{j=1}^{k} (n)_{k-1} A_n(\phi_j).
\end{aligned}$$

Dividing both sides by $(n)_k$ and rearranging terms, we get

$$A_n(\phi) = \frac{n}{n-k+1} A_n(f) A_n(g) - \frac{1}{n-k+1} \sum_{j=1}^{k} A_n(\phi_j),$$

Using above two results, theorem follows by induction. □