Independence and conditional independence : Examples & Exercises

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1. *An example where three events are pairwise independent, but not jointly independent*

   This example is to demonstrate that if there exist 3 events A, B and C such that

$$P(A \cap B) = P(A) \cdot P(B),$$
$$P(B \cap C) = P(B) \cdot P(C),$$
$$P(A \cap C) = P(A) \cdot P(C),$$

then it *need not necessarily be true* that $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$.

Consider an experiment that involves tossing 3 coins* C_0, C_1 and C_2. Clearly, the set of all possible outcomes is

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Let $\mathcal{F}$ be the set of *all* subsets of Ω (clearly this is a σ-algebra). Let

$$A \; - \; \text{be the event that coin } C_0 \text{ shows } H$$
$$B \; - \; \text{be the event that coin } C_1 \text{ shows } H$$
$$C \; - \; \text{be the event that coin } C_2 \text{ shows } H,$$

and let us consider the following assignment of probabilities:

Events	Probabilities
$\{HHH\}$	$\frac{1}{4}$
$\{HHT\}$	0
$\{HTH\}$	0
$\{HTT\}$	$\frac{1}{4}$
$\{THH\}$	0
$\{THT\}$	$\frac{1}{4}$
$\{TTH\}$	$\frac{1}{4}$
$\{TTT\}$	0

*A coin has two faces, named H, T. A toss of a coin results in one of the two faces.

Then, we have

$$\begin{aligned}
P(A) = \frac{1}{2},\ P(B) = \frac{1}{2},\ P(C) = \frac{1}{2},\\
P(A \cap B) = (\{HHH\}) + P(\{HHT\}) = \frac{1}{4} = P(A) \cdot P(B),\\
P(B \cap C) = P(\{THH\}) + P(\{HHH\}) = \frac{1}{4} = P(B) \cdot P(C),\\
P(A \cap C) = P(\{HHH\}) + P(\{HTH\}) = \frac{1}{4} = P(A) \cdot P(C),\ \text{but}\\
P(A \cap B \cap C) = P(\{HHH\}) = \frac{1}{4} \neq P(A) \cdot P(B) \cdot P(C).
\end{aligned}$$

2. *An example where three events are jointly independent, but not pairwise independent*

This example is to demonstrate that if there exist 3 events A, B and C such that

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C),$$

then it *need not necessarily be true* that

$$\begin{aligned}
P(A \cap B) = P(A) \cdot P(B),\\
P(B \cap C) = P(B) \cdot P(C),\\
P(A \cap C) = P(A) \cdot P(C),
\end{aligned}$$

i.e., at least one of these or all of these could be violated.

Consider the same example as above (with A, B and C being the same events as before), but now with the following assignment of probabilities:

Events	Probabilities
$\{HHH\}$	$\frac{1}{8}$
$\{HHT\}$	$\frac{2}{8}$
$\{HTH\}$	0
$\{HTT\}$	$\frac{1}{8}$
$\{THH\}$	0
$\{THT\}$	$\frac{1}{8}$
$\{TTH\}$	$\frac{3}{8}$
$\{TTT\}$	0

It is left as exercise to check that

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C),$$

but

$$P(A \cap B) \neq P(A) \cdot P(B),$$
$$P(B \cap C) \neq P(B) \cdot P(C),$$
$$P(A \cap C) \neq P(A) \cdot P(C).$$

3. *An example where two events are (unconditionally) independent but not conditionally independent (when conditioned on a third event)*

This example is to demonstrate that if there exist 3 events A, B and C such that

$$P(A \cap B) = P(A) \cdot P(B),$$

then it *need not necessarily be true* that

$$P(A \cap B|C) = P(A|C) \cdot P(B|C).$$

Consider two independent throws of a fair die.† Verify that

$$\Omega = \{(1,1), (1,2), \ldots, (1,6), (2,1), \ldots, (2,6), \ldots, (6,1), \ldots, (6,6)\}$$

Let $\mathcal{F}$ be the set of *all* subsets of Ω.
With this, we have specified what Ω and $\mathcal{F}$ are. Notice that we have also specified $P : \mathcal{F} \to [0,1]$ by saying that we are throwing a "fair" die.

Let A be the event that the first throw results in a 5. So,

$$A = \{(5,1), (5,2), \ldots, (5,6)\}$$

Let B be the event that the second throw results in a 3. Write out what B looks like. What is $A \cap B$?

Now, let us calculate $P(A \cap B)$. Recall that the two throws are independent of each other.

$$\begin{aligned} P(A \cap B) &= \frac{1}{36} \\ &= \frac{1}{6} \cdot \frac{1}{6} \\ &= P(A) \cdot P(B). \end{aligned}$$

†A die has six faces, numbered $1, 2, 3, 4, 5, 6$. A throw of a die results in one of the six faces. For a fair die, all faces are *equally likely* to result.

Thus, A and B are **independent**. Now, let C be the event that the sum of the results of the two throws is equal to 8. i.e.,

$$C = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}.$$

Clearly,

$$\begin{aligned} A \cap B \cap C &= \{(5,3)\} \\ A \cap C &= \{(5,3)\} \text{ and,} \\ B \cap C &= \{(5,3)\}. \end{aligned}$$

Now, conditioned on event C, are events A and B independent?

$$\begin{aligned} P(A \cap B|C) &= \frac{P(A \cap B \cap C)}{P(C)} \\ &= \frac{1/36}{5/36} \\ &= \frac{1}{5} \\ P(A|C) &= \frac{P(A \cap C)}{P(C)} \\ &= \frac{1/36}{5/36} \\ &= \frac{1}{5} \end{aligned}$$

Similarly, $P(B|C) = \frac{1}{5}$. Thus,

$$P(A \cap B|C) \neq P(A|C) \cdot P(B|C).$$

Therefore, conditioned on the event C, events A and B are **not *conditionally* independent**.

4. *An example where two events are conditionally independent (conditioned on a third event), but not independent by themselves unconditionally*

   This example is to demonstrate that if there exist 3 events A, B and C such that

   $$P(A \cap B|C) = P(A|C) \cdot P(B|C),$$

   then it *need not necessarily be true* that $P(A \cap B) = P(A) \cdot P(B)$.

   Consider two coins C_1 and C_2.
   Let the probability of C_1 resulting in H be 1/3 and that in T be 2/3.
   Let the probability of C_2 resulting in H be 1/4 and that in T be 3/4.
   Now, let C_0 be a fair coin. i.e., the probability of C_0 resulting in H is 1/2 and that

in T is $1/2$.

We go through the following procedure: Toss C_0. If it results in H, pick C_1 and toss it twice. If the toss of C_0 results in T, pick C_2 and toss it twice. Note that there are three coin tosses in all. Assume that all the tosses are independent of each other.

Let C be the event that the first toss (i.e., toss of C_0) results in H.
Let A be the event that the second toss results in H.
Let B be the event that the third toss results in H.

Now,

$$\begin{aligned} P(A \cap B|C) &= P(\text{second and third tosses result in H, given that coin } C_1 \text{ is picked}) \\ &= \frac{1}{3} \cdot \frac{1}{3}. \end{aligned}$$

Similarly we can see that

$$\begin{aligned} P(A|C) &= \frac{1}{3} \text{ and,} \\ P(B|C) &= \frac{1}{3}. \end{aligned}$$

Therefore,

$$P(A \cap B|C) = P(A|C) \cdot P(B|C).$$

So, conditioned on the event C, events A and B are ***conditionally*** **independent**.
Now,

$$\begin{aligned} P(A \cap B) &= P(A \cap B|C) \cdot P(C) + P(A \cap B|C^C) \cdot P(C^C) \\ &= \frac{1}{9} \cdot \frac{1}{2} + \frac{1}{16} \cdot \frac{1}{2} \\ &= \frac{25}{288}. \\ P(A) &= P(A|C) \cdot P(C) + P(A|C^C) \cdot P(C^C) \\ &= \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} \\ &= \frac{7}{24}. \end{aligned}$$

Similarly, $P(B) = \frac{7}{24}$. Therefore, $P(A) \cdot P(B) = \frac{24.5}{288}$.

$$P(A \cap B) \neq P(A) \cdot P(B).$$

Thus, A and B are **not independent**.

Miscellaneous exercise problems:

Assume $(\Omega, \mathcal{F}, P)$ is a probability space. All events in the questions below are subsets of Ω.

1. Show that given any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
2. Suppose event A is independent of event B. Show that the event A^C is independent of B. Also, show that A^C is independent of B^C.
3. Suppose $P(A_n) = 1$ for all $n = 1, 2, \ldots$. Show that $P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1$.
4. Suppose A and B be events such that $P(A) = 3/4$ and $P(B) = 1/3$. Show that
$$\frac{1}{12} \overset{(a)}{\leq} P(A \cap B) \overset{(b)}{\leq} \frac{1}{3}.$$
When does inequality (a) hold with equality? When does inequality (b) hold with equality? Give two examples, one where the first inequality (a) holds with equality and the other where the second inequality (b) holds with equality. Also produce an example where strict inequalities hold.