Random Variables and Distribution Functions : Exercises

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Assume $(\Omega, \mathcal{F}, P)$ is a probability space.

1. Let $X : \Omega \to \mathbb{R}$ be a mapping as defined in each of the cases below. For each case, verify that X is indeed a random variable with respect to $\mathcal{F}$ and construct the CDF of X.

   (a) $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\Omega}$, $P(\{H\}) = p$, $X = \mathbf{1}_{\{H\}}$ [†]

   (b) $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\Omega}$, $P(\{H\}) = p$, $X = 5 \cdot \mathbf{1}_{\{H\}}$.

   (c) $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\Omega}$, $P(\{H\}) = p$, $X = (-1)^{\mathbf{1}_{\{H\}}}$.

   (d) $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F}$ contains all singletons, ϕ and Ω, $P(\{\omega\}) = 1/6$ and $X(\omega) = \omega/2$ for all $\omega \in \Omega$.

   (e) $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\phi, \Omega\}$, $X(\omega) = 8$ for all $\omega \in \Omega$.

   (f) $\Omega = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$, $\mathcal{F} = 2^{\Omega}$, $P(\{\omega\}) = 1/9$ for all $\omega \in \Omega$, $X((a, b)) = a + b$ for all $a, b \in \{1, 2, 3\}$.

2. Let $X : \Omega \to \mathbb{R}$ be a random variable defined with respect to $\mathcal{F}$. Which of the following are valid CDFs of X? For each that is not valid, state at least one reason why. For each that is valid, find $P(\{\omega \in \Omega : X(\omega) > 5\})$ (this is written in short as $P(X > 5)$).

   (a) $F_X(x) = \begin{cases} \frac{e^{-x^2}}{4} & x < 0 \\ 1 - \frac{e^{-x^2}}{4} & x \geq 0. \end{cases}$

   (b) $F_X(x) = \begin{cases} 0 & x < 0 \\ 0.5 + e^{-x} & 0 \leq x < 3 \\ 1 & x \geq 3. \end{cases}$

   (c) $F_X(x) = \begin{cases} 0 & x \leq 0 \\ 0.5 + \frac{x}{20} & 0 < x \leq 10 \\ 1 & x > 10. \end{cases}$

[†] $\mathbf{1}_A$ stands for the indicator function of the set A. $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, 0 otherwise.

3. Let $X : \Omega \to \mathbb{R}$ be a random variable defined with respect to $\mathcal{F}$. Suppose that X has the following CDF:

$$F_X(x) = \begin{cases} 0 & x < -1 \\ 1-p & -1 \le x < 0 \\ 1-p+xp & 0 \le x \le 2 \\ 1 & x > 2, \end{cases}$$

where $p \in (0,1)$ is a fixed constant. Sketch this function, and find (i) $P(X = -1)$, (ii) $P(X = 0)$, and (iii) $P(X \geq 1)$.

4. If $X : \Omega \to \mathbb{R}$ is a random variable defined with respect to $\mathcal{F}$, and $a \in \mathbb{R}$ is a constant, show that $Y = aX$ is also a random variable with respect to $\mathcal{F}$ (hint: use the definition of a random variable).

5. Take $\Omega = \{1, 2, 3, 4\}$.
   (a) Construct a σ-algebra $\mathcal{F}$ of subsets of Ω such that
      (i) $\{1\} \in \mathcal{F}$ and $\{2\} \in \mathcal{F}$
      (ii) $\mathcal{F}$ is not the power set of Ω.
   (b) Construct a function $X : \Omega \to \mathbb{R}$ that is a random variable with respect to $\mathcal{F}$ constructed in part (a) above.

6. Take $\Omega = \mathbb{R}$. The Borel σ-algebra of subsets of $\mathbb{R}$, denoted as $\mathcal{B}(\mathbb{R})$ (or $\mathcal{B}$ in short), is the smallest σ-algebra containing all the open subsets of $\mathbb{R}$. Let $\mathcal{F} = \mathcal{B}$.
   (a) Show that $(-\infty, x] \in \mathcal{F}$ for all $x \in \mathbb{R}$.
   (b) Show that $[x, \infty) \in \mathcal{F}$ for all $x \in \mathbb{R}$.
   (c) Show that the sets $(1, 2), [1, 2], (1, 2]$ and $[1, 2)$ are present in $\mathcal{F}$ (more generally, for all $a, b \in \mathbb{R}$ such that $-\infty < a < b < \infty$, $(a, b), [a, b], (a, b]$ and $[a, b)$ are present in $\mathcal{F}$).
   (d) Show that $\{x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$.
   (e) Show that $\mathcal{F}$ contains all the closed subsets of $\mathbb{R}$.

Reference: The concept of Borel σ-algebra is taught in a course on measure theory. Prof. Krishna Jagannathan's NPTEL lectures on probability theory, available at `http://nptel.ac.in/courses/108106083/1`, explain these concepts in great detail. Interested students may go through these lectures at their convenience. Specifically, lectures $8, 9$ and 10 deal with Borel σ-algebra and Borel sets.