Random Variables - Examples & Exercises

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Assume $(\Omega, \mathcal{F}, P)$ is a probability space.

1. *If $X : \Omega \to \mathbb{R}$ is a random variable defined with respect to $\mathcal{F}$, and $a \in \mathbb{R}$ is any constant, show that $Y = aX$ is also a random variable with respect to $\mathcal{F}$.*

   $Y = aX$ is a function defined as $Y(\omega) = aX(\omega)$, $\omega \in \Omega$. Since X is given to be a random variable, the following statements are equivalent to (21):

$$\{\omega \in \Omega : X(\omega) \le y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}, \tag{1}$$
$$\{\omega \in \Omega : X(\omega) \ge y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}, \tag{2}$$
$$\{\omega \in \Omega : X(\omega) < y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}, \tag{3}$$
$$\{\omega \in \Omega : X(\omega) > y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}. \tag{4}$$

   In order to show that Y is a random variable, it suffices to show that

$$\{\omega \in \Omega : Y(\omega) \le x\} = \{\omega \in \Omega : aX(\omega) \le x\} \in \mathcal{F} \text{ for all } x \in \mathbb{R}. \tag{5}$$

   (a) Case 1: Suppose $a = 0$. Then,

$$\begin{aligned}\{\omega \in \Omega : Y(\omega) \le x\} &= \{\omega \in \Omega : aX(\omega) \le x\} \\ &= \{\omega \in \Omega : 0 \le x\} \\ &= \begin{cases} \phi, & x < 0 \\ \Omega, & x \ge 0. \end{cases}\end{aligned} \tag{6}$$

   From the above description, it is clear that $\{\omega \in \Omega : Y(\omega) \le x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$. Thus, $Y = aX$ is a random variable when $a = 0$.

   (b) Case 2: Suppose $a > 0$. Then, for any $x \in \mathbb{R}$,

$$\begin{aligned}\{\omega \in \Omega : Y(\omega) \le x\} &= \{\omega \in \Omega : aX(\omega) \le x\} \\ &= \left\{\omega \in \Omega : X(\omega) \le \frac{x}{a}\right\} \in \mathcal{F}\end{aligned} \tag{7}$$

   since (1) holds with $y = \frac{x}{a}$. Thus, $Y = aX$ is a random variable for any $a > 0$.

   (c) Case 3: Suppose $a < 0$. Then, for any $x \in \mathbb{R}$,

$$\begin{aligned}\{\omega \in \Omega : Y(\omega) \le x\} &= \{\omega \in \Omega : aX(\omega) \le x\} \\ &= \left\{\omega \in \Omega : X(\omega) \ge \frac{x}{a}\right\} \in \mathcal{F}\end{aligned} \tag{8}$$

since (2) holds with $y = \frac{x}{a}$. Thus, $Y = aX$ is a random variable for any $a < 0$.

2. *If X and Y are two random variables defined with respect to $\mathcal{F}$, show that $X + Y$ is also a random variable with respect to $\mathcal{F}$.*

Since X and Y are given to be random variables, by definition,

$$\{\omega \in \Omega : X(\omega) < y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}, \tag{9}$$

$$\{\omega \in \Omega : Y(\omega) < y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}. \tag{10}$$

In order to show that $X + Y$ is a random variable, it suffices to show that

$$\{\omega \in \Omega : X(\omega) + Y(\omega) < x\} \in \mathcal{F} \text{ for all } x \in \mathbb{R}. \tag{11}$$

Fix an arbitrary $x \in \mathbb{R}$. Then, $X(\omega) + Y(\omega) < x$ implies that there exists a rational number $q \in \mathbb{Q}$ such that $X(\omega) < q$ and $Y(\omega) < x - q$. Conversely, if there exists a rational number $q \in \mathbb{Q}$ such that $X(\omega) < q$ and $Y(\omega) < x - q$, then this implies that $X(\omega) + Y(\omega) < x$. By translating the words "there exists" and "and" into union and intersection of sets respectively, we get that

$$\{\omega \in \Omega : X(\omega) + Y(\omega) < x\} = \bigcup_{q \in \mathbb{Q}} \underbrace{\left(\underbrace{\{\omega \in \Omega : X(\omega) < q\}}_{\in \mathcal{F} \text{ from (9) with } y=q} \cap \underbrace{\{\omega \in \Omega : Y(\omega) < x - q\}}_{\in \mathcal{F} \text{ from (10) with } y=x-q} \right)}_{\in \mathcal{F} \text{ since intersection of two events in } \mathcal{F} \text{ belongs to } \mathcal{F}}$$

belongs to $\mathcal{F}$ since the union over $z \in \mathbb{Z}$ is a countable union, and countable union of events in $\mathcal{F}$ belongs to $\mathcal{F}$ by the property that $\mathcal{F}$ is a σ-algebra. Thus, $X + Y$ is a random variable.

<u>Note 1</u>: In the above analysis, it is crucial that X and Y are both defined with respect to $\mathcal{F}$. In other words, if X is defined with respect to $\mathcal{F}$ and Y is defined with respect to a different σ-algebra $\mathcal{G}$, then $X + Y$ is not a meaningful definition.

<u>Note 2</u>: The above problem can also be solved using the fact that a continuous function of random variables is a random variable.

3. *If X and Y are random variables defined with respect to $\mathcal{F}$, show that* $\max\{X, Y\}$ *is also a random variable with respect to $\mathcal{F}$.*

Since X and Y are given to be random variables, by definition,

$$\{\omega \in \Omega : X(\omega) \le y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}, \tag{12}$$

$$\{\omega \in \Omega : Y(\omega) \le y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}. \tag{13}$$

We need to show that

$$\{\omega \in \Omega : \max\{X(\omega), Y(\omega)\} \le x\} \in \mathcal{F} \text{ for all } x \in \mathbb{R}. \tag{14}$$

Fix an arbitrary $x \in \mathbb{R}$. Then, $\max\{X(\omega), Y(\omega)\} \le x$ implies that $X(\omega) \le x$ and $Y(\omega) \le x$, and the converse is also true. Thus,

$$\{\omega \in \Omega : \max\{X(\omega), Y(\omega)\} \le x\} = \underbrace{\{\omega \in \Omega : X(\omega) \le x\}}_{\in \mathcal{F} \text{ from (12) with } y=x} \cap \underbrace{\{\omega \in \Omega : Y(\omega) \le x\}}_{\in \mathcal{F} \text{ from (13) with } y=x} \tag{15}$$

belongs to $\mathcal{F}$ since intersection of two events in a $\mathcal{F}$ belongs to $\mathcal{F}$ by the property that $\mathcal{F}$ is a σ-algebra. Hence $\max\{X, Y\}$ is a random variable.

4. *Show that if X is a random variable defined with respect to $\mathcal{F}$, then X^2 is also a random variable defined with respect to $\mathcal{F}$.*

   Since X is given to be a random variable, by definition,

$$\{\omega \in \Omega : X(\omega) \le y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}, \tag{16}$$
$$\{\omega \in \Omega : X(\omega) \ge y\} \in \mathcal{F} \text{ for all } y \in \mathbb{R}. \tag{17}$$

   We need to show that

$$\{\omega \in \Omega : (X(\omega))^2 \le x\} \in \mathcal{F} \text{ for all } x \in \mathbb{R}. \tag{18}$$

   Clearly, since $(X(\omega))^2$ is a non-negative real number, $\{\omega \in \Omega : (X(\omega))^2 \le x\} = \phi$ for all $x < 0$. Fix an arbitrary $x \ge 0$. Then,

$$\begin{aligned}\{\omega \in \Omega : (X(\omega))^2 \le x\} &= \{\omega \in \Omega : |X(\omega)| \le \sqrt{x}\} \\ &= \{\omega \in \Omega : -\sqrt{x} \le X(\omega) \le \sqrt{x}\} \\ &= \underbrace{\{\omega \in \Omega : -\sqrt{x} \le X(\omega)\}}_{\in \mathcal{F} \text{ from (17) with } y=-\sqrt{x}} \cap \underbrace{\{\omega \in \Omega : X(\omega) \le \sqrt{x}\}}_{\in \mathcal{F} \text{ from (16) with } y=\sqrt{x}} \qquad (19)\end{aligned}$$

   belongs to $\mathcal{F}$ since intersection of two events in $\mathcal{F}$ belongs to $\mathcal{F}$ by the property that $\mathcal{F}$ is a σ-algebra. Hence X^2 is a random variable.

5. Let $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra of subsets of $\mathbb{R}$. If $B \in \mathcal{B}$, then B is known as a *Borel set.* Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on $\mathbb{R}$. Then, f is said to be a **Borel measurable function** if:

$$f^{-1}(B) \in \mathcal{B} \text{ for all } B \in \mathcal{B}, \tag{20}$$

   i.e., if the inverse image (under f) of every Borel set is a Borel set.

6. *If $X : \Omega \to \mathbb{R}$ is a random variable defined with respect to $\mathcal{F}$ and $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, show that $f(X) : \Omega \to \mathbb{R}$ is also a random variable with respect to $\mathcal{F}$.*

   Since X is a random variable, by definition,

$$X^{-1}(A) \in \mathcal{F} \text{ for every } A \in \mathcal{B}, \tag{21}$$

and since f is Borel measurable,

$$f^{-1}(B) \in \mathcal{B} \text{ for all } B \in \mathcal{B}. \tag{22}$$

In order to show that $g = f(X)$ is a random variable, we need to show that

$$g^{-1}(B) \in \mathcal{F} \text{ for every } B \in \mathcal{B}. \tag{23}$$

Fix an arbitrary $B \in \mathcal{B}$. Then,

$$\begin{aligned} g^{-1}(B) &= (f(X))^{-1}(B) \\ &= X^{-1}(f^{-1}(B)) \\ &= X^{-1}(A) \\ &\in \mathcal{F}, \end{aligned} \tag{24}$$

where $A = f^{-1}(B) \in \mathcal{B}$ from (22) since f is Borel measurable, and $X^{-1}(A) \in \mathcal{F}$ from (21) since X is a random variable.

Remark: Every continuous function is Borel measurable. Hence, if $X : \Omega \to \mathbb{R}$ is a random variable defined with respect to $\mathcal{F}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f(X) : \Omega \to \mathbb{R}$ is also a random variable with respect to $\mathcal{F}$. Thus, for example, if X is a random variable, so are $|X|$, e^X, X^2, $\sin(X)$, $aX + b$ (for any $a, b \in \mathbb{R}$), etc. On similar lines, if X and Y are random variables defined with respect to $\mathcal{F}$, then so are $X + Y$, $X - Y$, $\log(|X + Y|)$, etc.

Miscellaneous exercises:

Assume $(\Omega, \mathcal{F}, P)$ is a probability space, and all random variables defined below are functions on Ω.

1. If X and Y are random variables defined with respect to $\mathcal{F}$, show that the following are also random variables defined with respect to $\mathcal{F}$ (**do not** use the fact that continuous functions of random variables are random variables):
   (i) $|X|$, $|Y|$
   (ii) $X - Y$
   (iii) XY
   (iv) $\min\{X, Y\}$
   (v) $X_+ := \max\{X, 0\}$, $X_- := -\min\{X, 0\}$
   (vi) $|X - Y|$.
2. If X and Y are random variables defined with respect to $\mathcal{F}$, show that
$$\{\omega \in \Omega : X(\omega) = Y(\omega)\} \in \mathcal{F}.$$
3. Let $X : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a random variable defined with respect to $\mathcal{F}$. Then, show that $\{\omega \in \Omega : |X(\omega)| = \infty\} \in \mathcal{F}$ (in this example, X is allowed to take the values $-\infty$ and $+\infty$).

4. Prove, by induction, that for any $n \geq 1$, if $X_1, \ldots, X_n$ are random variables, all defined with respect to $\mathcal{F}$, then the following are also random variables with respect to $\mathcal{F}$:
   (a) $\frac{X_1 + \ldots + X_n}{n}$
   (b) $\frac{X_1 + \ldots + X_n}{\sqrt{n}}$.

5. Let X be a random variable defined with respect to $\mathcal{F}$, and suppose $X_1, X_2, \ldots$ is a sequence of random variables, all defined with respect to $\mathcal{F}$. Then, show that for any $\epsilon > 0$,
$$\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon\} \in \mathcal{F} \text{ for all } n \geq 1.$$