Convergence of Sequence of Random Variables - Some exercises

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Prepared by - Karthik and Sahasranand

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1. Let $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}([0,1]), \lambda)$, where λ denotes the standard Lebesgue measure on $\mathbb{R}$.
   (a) Let $X_n = n \cdot 1_{[0,\frac{1}{n}]}$ (here, 1_A denotes the indicator function of the set A). Sketch the cdf of X_n, and show that $X_n \xrightarrow{d} 0$.
   (b) Show that X_n's as defined above are not independent.
2. If $\sum_{n=1}^{\infty} E[|X_n|^p] < \infty$ for some $p > 0$, then show that $X_n \xrightarrow{a.s.} 0$.
3. Let X_n's be random variables such that $P(X_n = 0) = \frac{1}{n} = 1 - P(X_n = 1)$ for all n, and let X be such that $P(X = 1) = 1$. Prove that X_n converges to X in distribution and in probability (prove both separately. Do not use the fact that convergence in probability implies that in distribution).
4. Let $(\Omega, \mathcal{F}, P) = ((0,1], \mathcal{B}([0,1]), \lambda)$, where λ denotes the standard Lebesgue measure on $\mathbb{R}$. In each of the cases below, identify the limit and the notion(s) of convergence to this limit.
   (a) $X_n(\omega) = n^2\omega \cdot 1_{(0,\frac{1}{n})}(\omega)$
   (b) $X_n(\omega) = n\omega - \lfloor n\omega \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x
   (c) $X_n(\omega) = n \cdot \omega^n$.
5. (*Convergence in distribution to convergence in probability*) Suppose $X_n \xrightarrow{d} c$, where $c \in \mathbb{R}$ is a constant. Then, show that $X_n \xrightarrow{i.p.} c$.
6. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $X_1, X_2, \ldots$ be a sequence of independent real-valued random variables defined on $(\Omega, \mathcal{F})$. Let A be a Borel set such that $P(X_k \in A) = p$ for all $k \geq 1$, and let $Y_1, Y_2, \ldots$ be another sequence of random variables defined as

$$Y_n := \frac{1}{n} \sum_{k=1}^{n} 1_{\{X_k \in A\}}.$$

   In other words, the random variable $(n \cdot Y_n)$ counts the number of times $X_k \in A$ for $1 \leq k \leq n$.
   (a) Show that Y_n converges to p in probability (do not use weak law of large numbers. Show explicitly using the definition of convergence in probability).
   (b) Does Y_n converge to p in the mean-squared sense? Justify your answer.
7. Let $X_1, X_2, \ldots$ be iid Exp(1) random variables. Define $Y_n := \max\{X_1, \ldots, X_n\}$.

(a) Compute the cdf of Y_n.

(b) Let $a, b \in \mathbb{R}$ such that $0 < a < 1 < b$. Show that

$$\begin{aligned} P(Y_n \leq a \log(n)) &\longrightarrow 0 \text{ as } n \to \infty \\ P(Y_n \leq b \log(n)) &\longrightarrow 1 \text{ as } n \to \infty. \end{aligned}$$

(c) Deduce that $\dfrac{Y_n}{\log(n)} \xrightarrow{d} 1$.

8. (*Convergence in distribution need not imply convergence in probability*) Let X be a Ber(0.5) random variable. For each $n \geq 1$, let $Y_n = X$. Let $Y = 1 - X$.

   (a) Show that $Y_n \xrightarrow{d} Y$.

   (b) Show that Y_n does not converge to Y in probability.