Transformation of random variables - Some exercises

$E2-202$ - Random Processes, Fall 2017, ECE, IISc.
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September 16, 2017

1. Let X be a discrete random variable. Calculate the pmf of Y in all the cases below:

   (a) $X \sim \text{unif}\{-n, -n+1, \ldots, -1, 0, 1, \ldots, n\}$ and $Y = |X|$

   (b) $X \sim \text{Bin}(n, p)$ and $Y = n - X$. What is the distribution of Y?

   (c) $X \sim \text{Poi}(\lambda)$ and $Y = 2X$

   (d) $X \sim \text{Geo}(p)$ and $Y = \frac{1}{X+1}$.

2. Let $U \sim \text{unif}(0, 1)$.

   (a) What is the pmf of $\lfloor nU \rfloor + 1$ where n is a fixed positive integer?
   Note: $\lfloor a \rfloor$ stands for the largest integer smaller than or equal to a.

   (b) For $0 < q < 1$, $X = \left\lfloor \log_q U \right\rfloor + 1$. Show that $X \sim \text{Geo}(1-q)$.

3. Let the pdf of X be

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Calculate $\mathbb{P}[X \ge 0.4 | X \le 0.8]$.

   (b) Let $Y = -\log X$. Calculate the pdf of Y.

4. Let $X \sim \text{Exp}(\lambda)$ and $Z = \min\{X, 3\}$. Calculate the pdf of Z.

5. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = e^X$. Calculate the pdf of Y (Y is called a log Normal random variable).

6. Let U and V be random variables with joint pdf

$$f_{U,V}(u, v) = \begin{cases} u + v, & 0 \le u \le 1,\ 0 \le v \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Let $X = U^2$, $Y = U(1+V)$. Calculate $f_{X,Y}(x, y)$.

7. Let $\Theta \sim \text{unif}(-\pi/2, \pi/2)$. Let $Y = \tan\Theta$. Verify that the pdf of Y is given by

$$f_Y(y) = \frac{1}{\pi(y^2+1)}, \; y \in (-\infty, \infty).$$

Y is called a Cauchy random variable.

   (a) Does $E[Y]$ exist?
   (b) Does $E[Y^2]$ exist?
   (c) Does $\text{Var}[Y]$ exist?

8. Let X_1, X_2 be independent zero mean Normals with variance σ^2. Define

$$R = \sqrt{X_1^2 + X_2^2}, \quad \Theta = \tan^{-1}\left(\frac{X_2}{X_1}\right).$$

   (a) Compute the densities of R and Θ.
   (b) Are R and Θ independent?
   (c) Let $S = X_1^2 + X_2^2 = R^2$. What is the distribution of S?

9. Let $X_1 \sim \text{Exp}(\lambda_1)$, $X_2 \sim \text{Exp}(\lambda_2)$ be independent. Let $R = \frac{X_1}{X_2}$. Compute the pdf of R.