Moment generating function, Characteristic function and Gaussian random vectors - Some exercises

$E2-202$ - Random Processes, Fall 2017, ECE, IISc.
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1. Show that for $X \sim \mathcal{N}(0, \sigma^2)$ and any $\epsilon > 0$,

   (a) $P[X \geq \epsilon] \leq e^{\frac{-\epsilon^2}{2\sigma^2}}$

   (b) $P[|X| \geq \epsilon] \leq 2e^{\frac{-\epsilon^2}{2\sigma^2}}$.

   Hint: Calculate the MGF of $\mathcal{N}(0, \sigma^2)$; use Markov's inequality.

2. Calculate the MGF $M_X(t)$ of $X \sim \mathcal{N}(\mu, \sigma^2)$. Verify that

   (a) $M'_X(0) = E[X] = \mu$, and

   (b) $M''_X(0) = E[X^2] = \sigma^2 + \mu^2$.

3. Compute the MGF of the following distributions.

   (a) $\mathrm{Ber}(p)$, $0 \leq p \leq 1$

   (b) $\mathrm{Bin}(n, p)$, $0 \leq p \leq 1, n \in \mathbb{Z}^+$

   (c) $\mathrm{Geo}(p)$, $0 \leq p \leq 1$

   (d) $\mathrm{Poi}(\lambda)$, $\lambda > 0$

   (e) $\mathrm{Exp}(\lambda)$, $\lambda > 0$

4. Compute the characteristic function of the following distributions.

   (a) $\mathrm{Ber}(p)$, $0 \leq p \leq 1$

   (b) $\mathrm{Bin}(n, p)$, $0 \leq p \leq 1, n \in \mathbb{Z}^+$

   (c) $\mathrm{Poi}(\lambda)$, $\lambda > 0$

   (d) $\mathrm{Unif}(-a, a)$, $a > 0$

   (e) $\mathcal{N}(0, 1)$

5. (a) Verify that for any $a, b \in \mathbb{R}$,

$$\Phi_{aX+b}(\omega) = e^{jb\omega} \cdot \Phi_X(a\omega) \quad \forall \omega \in \mathbb{R}.$$

(b) Show that
$$\overline{\Phi_X(\omega)} = \Phi_X(-\omega) \quad \forall \omega \in \mathbb{R}.$$
Note: For a complex number $z = x + jy$, $\bar{z}$ denotes its complex conjugate, i.e., $\bar{z} = x - jy$.

(c) Show that $\Phi_X(\omega)$ is a real-valued function of ω if and only if the distribution of X is symmetric (i.e., X and $-X$ have the same distribution).

6. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Using the characteristic function of X, derive the distribution of $Y = \dfrac{X - \mu}{\sigma}$.

7. Let $X = (X_1, X_2, X_3)$ be a zero mean Gaussian random vector with covariance matrix
$$K = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

(a) Write down the distributions of X_1, X_2 and X_3.

(b) Does a joint density exist for X? If not, what is the relationship between X_1, X_2 and X_3?

(c) Does a joint density exist for $X' = (X_1, X_2)$? If not, what is the relationship between X_1 and X_2?

(d) Does a joint density exist for $Y' = (X_2, X_3)$? If not, what is the relationship between X_2 and X_3?

(e) Does a joint density exist for $Z' = (X_1, X_3)$? If not, what is the relationship between X_1 and X_3?

8. *Rotation of a joint normal distribution yielding independence* –
Let X be a Gaussian random vector with
$$E[X] = \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \quad \mathrm{Cov}(X) = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

(a) Compute the pdf of X explicitly.

(b) Find a vector b and an orthogonal matrix U such that the vector $Y = [Y_1\ Y_2]^T$ defined by $Y = U(X - b)$ is a mean zero Gaussian vector such that Y_1 and Y_2 are independent.

Remark – Indeed, a set of n random variables $Z_1, \ldots, Z_n$ is jointly Gaussian if and only if:

(a) there exists an integer $l \geq 1$,

(b) there exists an $(n \times l)$ matrix A and a vector $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, and

(c) there exist iid $\mathcal{N}(0, 1)$ random variables $W_1, \ldots, W_l$

such that

$$\begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} = A \cdot \begin{pmatrix} W_1 \\ \vdots \\ \vdots \\ W_l \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

This can be used as an alternative definition for jointly Gaussian random variables, and it can be shown that this definition is equivalent to the ones stated in class. Notice here that l can be greater than, equal to or less than n.