Joint and Conditional Distributions - Some exercises

$E2-202$ - Random Processes, Fall 2017, ECE, IISc.
Prepared by - Karthik and Sahasranand

Sep. 6 2017

1. Let X and Y have a joint pdf
$$f(x,y) = x + y, \;\; 0 \le x \le 1, 0 \le y \le 1.$$
Are X and Y independent?

2. X and Y are jointly continuous with joint pdf
$$f(x) = \begin{cases} cx^2 + \frac{xy}{3}, \text{ if } 0 \le x \le 1, 0 \le y \le 2 \\ 0, \text{ otherwise.} \end{cases}$$
   (a) Find c.
   (b) Are X and Y independent?
   (c) Calculate $\mathrm{Cov}(X,Y)$.

3. Let X and Y be independent random variables distributed uniformly on $[0,1]$. Let $U = \min\{X,Y\}$ and $V = \max\{X,Y\}$. Calculate $\mathrm{Cov}(U,V)$.

4. Suppose $X_1, X_2, \ldots$ are independent random variables distributed uniformly on $(0,1)$.
   (a) Let $N = \min\{n \ge 1 : X_{n+1} > X_n\}$. Find the CDF of N and $E[N]$.
   (b) Let $N = \min\{n \ge 1 : \sum_{k=1}^{n} X_k \ge 1\}$. Find the CDF of N and $E[N]$.

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $X : \Omega \to [0, \infty)$ be a nonnegative random variable defined with respect to $\mathcal{F}$. If $E[X] = 0$, then show that $P(X = 0) = 1$.

6. Suppose (U,V) is uniformly distributed over the square with corners $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, and let $X = UV$. Find the CDF and pdf of X.

7. (a) Give an example for a distribution whose mean and variance are finite.
   (b) Produce an example for a distribution whose mean is finite but variance is infinite.
   (c) Produce an example for a distribution whose mean and variance are infinite (note that you are required to produce an example such that mean exists and equals $+\infty$).