E2 202 Quiz 4 Oct 28, 2017

Name: ______________________ SR No: ______________________

Question:	1	Total
Points:	5	5
Score:		

1. Let $(X_k : k \in \mathbb{N})$ be a sequence of independent and identically distributed random variables with $P(X_k = 1) = P(X_k = -1) = 0.5$ for all k. For each $n \in \mathbb{N}$, define $P_n = \prod_{k=1}^{n} X_k$.
   (a) Evaluate the mean and autocorrelation of the process $(P_n : n \in \mathbb{N})$. (3)
   (b) Is the process $(P_n : n \in \mathbb{N})$ wide-sense stationary? Justify your answer. (2)

Summary for E2-202 Course

1. $(\Omega, \mathcal{F})$ measurable space ; Ω - non-empty but otherwise ~~no~~ arbitrary
$\mathcal{F}$ - σ-algebra of subsets of Ω
(i) $\Omega \in \mathcal{F}$
(ii) $A \in \mathcal{F} \Leftrightarrow A^c \in \mathcal{F}$
(iii) $A_1, A_2 \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$P : \mathcal{F} \to [0,1]$ prob. measure
(i) $P(\Omega) = 1$
(ii) $A_1, A_2, \ldots$ mutually (pairwise) disjoint
$$P\left(\biguplus_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$(\Omega, \mathcal{F}, P)$ - probability space.

Note : If $A \in \mathcal{F}$ and $B \subseteq A$, then it need not be the case that $B \in \mathcal{F}$.
That is, not every subset of a set should belong to $\mathcal{F}$.

2. Some corollaries from above axioms :

- $A \subseteq B \Rightarrow P(A) \leq P(B)$, equality iff $P(B|A) = 0$.
  Converse not true, i.e., $P(A) \leq P(B) \not\Rightarrow A \subseteq B$.
- $P(\phi) = 0$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Finite additivity : $A_1, \ldots, A_n$ pairwise disjoint. Then, $P\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$.

3. Continuity of prob. measure P :

(i) If $A_1 \subseteq A_2 \subseteq \ldots$, $A_i \in \mathcal{F}$, then
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} P(A_i)$$

(ii) If $A_1 \supseteq A_2 \supseteq \ldots$, $A_i \in \mathcal{F}$, then
$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim P(A_i).$$

3. **Conditional probability** : ~~Two~~ events A and B given. Suppose $P(B)>0$.

$\cancel{P(A\cap B) = P(A)}$

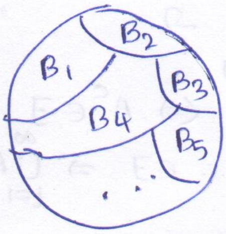
$$P(A|B) := \frac{P(A\cap B)}{P(B)}.$$

This def^n makes sense only if $P(B)>0$. Otherwise, if $P(B)=0$, all we can say is $P(A\cap B) \le P(B) = 0$ $\Leftrightarrow P(A\cap B)=0$.

Note : $A|B$ is not an event. The above def^n means that for fixed B s.t. $P(B)>0$,

$$Q(\cdot) = P(\cdot|B) : \mathcal{F} \longrightarrow [0,1] \text{ is a probability measure}$$

Law of total probability ; Partition Ω into $B_1, B_2, \ldots,$ i.e.,



$$\Omega = \biguplus_{i=1}^{\infty} B_i, \quad B_i \cap B_j = \phi \quad \forall\, i \neq j.$$

Then, for any event $A \in \mathcal{F}$,

$$\cancel{P(A) = \sum_{i=1}^{\infty} P(A|B_i)\, P(B_i}$$

$$\begin{aligned} P(A) &= P(A\cap \Omega) \\ &= P\Big(A \cap \biguplus_{i=1}^{\infty} B_i\Big) \\ &= P\Big(\biguplus_{i=1}^{\infty} A\cap B_i\Big) \\ &= \sum_{i=1}^{\infty} P(A\cap B_i) \end{aligned}$$

Further, if $P(B_i)>0$ $\forall\, i$, then

$$P(A) = \sum_{i=1}^{\infty} P(A|B_i)\, P(B_i)$$

Bayes rule :

$$P(B_i|A) = \frac{P(A|B_i)\, P(B_i)}{P(A)}.$$

when $P(B_i)>0$ $\forall\, i$ and $P(A)>0$.

(This is just a restatement of saying

$$\begin{aligned} P(A)\cdot P(B_i|A) &= P(A\cap B_i) \\ &= P(B_i)\, P(A|B_i). \Big) \end{aligned}$$

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   (a) Evaluate the mean and autocorrelation of the process $(P_n : n \in \mathbb{N})$. **(3)**
   (b) Is the process $(P_n : n \in \mathbb{N})$ wide-sense stationary? Justify your answer. **(2)**

4. Random variables:

$(\Omega, \mathcal{F})$ given:

$X : \Omega \to \mathbb{R}$ is called a rv (or a measurable function) iff:

$\forall\ B \in \mathcal{B}(\mathbb{R}),\ A = X^{-1}(B) \in \mathcal{F}$.

When we consider $\mathbb{R}$, associated σ-algebra is the Borel σ-algebra of subsets of $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$. How we construct $\mathcal{B}(\mathbb{R})$ is: collect all open subsets of $\mathbb{R}$, & build the smallest σ-algebra starting from it

$$\begin{array}{ccc} (\Omega, \mathcal{F}) & \xrightarrow{X} & \mathbb{R}, \mathcal{B}(\mathbb{R}) \\ \uparrow & & \downarrow \\ A & \xleftarrow{X^{-1}(B)} & B \end{array}$$

- The def^n of rv depends on what $\mathcal{F}$ is. A func^n which is a rv wrt a σ-algebra need not be a rv wrt another, perhaps different, σ-algebra.

5. CDF:

$$(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$$

P induces a prob. measure P_X on $\mathbb{R}$ through X.

P and P_X are nicely related as:

$$P_X\big((-\infty, x]\big) = P\big(\omega : X(\omega) \le x\big).$$

This is called the cdf of X, denoted $F_X(x)$.

Thus,

$$F_X(x) = P(X \le x) \quad \forall\ x \in \mathbb{R}$$

is a func^n on $\mathbb{R}$, taking values in $[0,1]$

$$\big(F_X : \mathbb{R} \to [0,1]\big).$$

- Properties of cdf:
  - Right continuous $(F(x+h) \to F(x)$ as $h \downarrow 0)$
  - Non-decreasing $(F(x) \le F(y)$ if $x < y)$
  - $\lim_{x\to\infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$.
- CDF always exists!

---

6. Pmf and pdf

Discrete rvs: $X : \Omega \to \mathbb{R}$ is discrete if it takes at most countably many values. We define pmf for X as

$$p_X(x) = P(X=x).$$

pmf sums up to 1, i.e., $\sum_x p_X(x) = 1$.

Continuous rvs: $X : \Omega \to \mathbb{R}$ is continuous if its cdf is continuous. We define pdf for X as

$$f_X(x) = \left.\frac{d}{dt} F_X(t)\right|_{t=x}.$$

pdf integrates to 1, i.e; $\int_{\mathbb{R}} f_X(x)\,dx = 1$.

Note: pdf need not always exist.

Similar notions extend to random vectors. There we have joint cdf and joint pmf/pdf.

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7. Independence:

(i) Of events: $A, B \in \mathcal{F}$ are indep iff $P(A \cap B) = P(A)\,P(B)$

(ii) Of rvs: Two rvs $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are indep. iff $\forall$ Borel sets C and D, we have

$$P(X\in C,\ Y \in D) = P(X \in C)\,P(Y\in D).$$

indep & cond^n^l indep.

Instead of checking $\forall\ C, D \in \mathcal{B}(\mathbb{R})$, it suffices to check for $(-\infty, x]$ and $(-\infty, y]$ $\forall\ x, y \in \mathbb{R}$:

$$X \perp\!\!\!\perp Y \iff P(X \le x,\ Y \le y) = P(X \le x)\,P(Y \le y) \quad \forall\ x, y \in \mathbb{R}.$$

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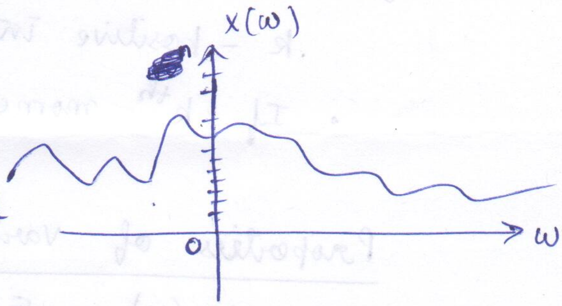
8. Expectations as Lebesgue integrals

(i) For simple rvs: $X : \Omega \to \mathbb{R}$ is simple if it takes only finitely many values, say $a_1, \ldots, a_n \in \mathbb{R}$, $n < \infty$. For such a rv, expectation is defined as

$$E[X] = \sum_{i=1}^{n} a_i \, P(X = a_i).$$

(ii) For non-negative rvs:

Given $X \geq 0$, perform a quantization into $n \cdot 2^n$ levels. Call this quantized rv as X_n, i.e.,



$$X_n(\omega) = \sum_{i=1}^{n \cdot 2^n} \frac{i-1}{2^n} \cdot \mathbb{1}_{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)}(X_n(\omega)) + n \, \mathbb{1}_{(X_n(\omega) \geq n)}.$$

Then, $X_n(\omega) \uparrow X(\omega)$ $\forall \omega \in \Omega$. Define

$$E[X] = \lim_{n \to \infty} E[X_n].$$

(iii) For arbitrary rvs:

Suppose $X : \Omega \to \mathbb{R}$ is a rv. Define

$$X_+ = \max\{0, X\}, \quad X_- = -\min\{0, X\}.$$

Then, EX is defined as

$EX = EX_+ - EX_-$ provided the RHS is not $\infty - \infty$. If RHS is $\infty - \infty$, then EX is undefined (or EX doesn't exist).

Note: • EX exists iff ~~EX~~

$$\min\{EX_+, EX_-\} < \infty.$$

• EX is finite iff $E|X| < \infty$.

Properties of expectation

1. $E[aX + bY] = a\,EX + b\,EY$ (linearity) (only for sums of finite # rvs, not infinite, i.e., $E\left[\sum_{k=1}^{\infty} a_k X_k\right] \neq \sum_{k=1}^{\infty} a_k E[X_k]$ in general)

2. If $X \le Y$, $EX \le EY$.

3. Suppose $X \ge 0$ is a non-negative rv. Then

   (i) $EX = 0 \Rightarrow P(X=0) = 1$.

   (ii) $EX < \infty \Rightarrow P(X<\infty) = 1 \longrightarrow$ this is sometimes useful in the context of stopping times, to show that a stopping time is finite with prob. 1.

4. $E[XY] = E[X]\,E[Y]$ $\longleftarrow$ X and Y are uncorrelated if

4. If X and Y are independent, then X & Y are uncorrelated.

$$X \perp\!\!\!\perp Y \Rightarrow X \text{ \& } Y \text{ uncorrelated}$$

$\overset{?}{\Leftarrow}$ Not always.

5. X rv, k – positive integer
   - If k^{th} moment is finite, all lower order moments are finite.

Properties of variance

$$Var(X) = E\left[(X-EX)^2\right] \overset{\text{if } EX<\infty}{=} EX^2 - (EX)^2$$

- For any 2 rvs X and Y,

$$Var(X+Y) = Var(X) + Var(Y) + 2\,cov(X,Y)$$

- If X and Y are uncorrelated,

$$Var(X+Y) = Var(X) + Var(Y).$$

- $Var(aX) = a^2\,Var(X)$

- Cauchy-Schwarz

$$Cov(X,Y) \le \sqrt{Var(X)\,Var(Y)}$$

Define $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$ — Correlation coefficient.

Then, Cauchy-Schwarz says

$$|\rho(X,Y)| \le 1$$

Equality implies $\exists\, a, b \in \mathbb{R}$ s.t $X = aY + b$.

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9. Some inequalities :

- Markov (only for non-neg rvs) Suppose $X \geq 0$. Then,

$$P(X>t) \leq \frac{EX}{t} \quad \forall\ t>0.$$

- Chebyshev (for any rv)

$$P(|X-EX|>t) \leq \frac{Var(X)}{t^2}.$$

- Chernoff bound (for any rv)

$$P(X>a) = P(tX > ta) \quad \forall\ t>0$$

$$= P(e^{tX} > e^{ta})$$

$$\leq \frac{E[e^{tX}]}{e^{ta}} \text{ (by Markov)}, \quad \forall\ t>0$$

$$\Rightarrow P(X>a) \leq \inf_{t>0} \frac{E[e^{tX}]}{e^{ta}}.$$

10. MGF and characteristic func^n

$$M_X(t) = E[e^{tX}]$$ (expect. always exists since $e^{tX} > 0$), $t \in \mathbb{R}$.

If $M_X(t) < \infty$ $\forall\ t \in (-\delta, \delta)$ for some $\delta > 0$, then all moments of X exist.

- Properties : (i) $M_X(0) = 1$.
  (ii) If all moments exist, then

$$E[X^n] = \frac{d^n}{} M_X(t) \Big|$$

- If $m_X(t) = m_Y(t) \quad \forall\, t \in \mathbb{R}$, then $F_X(x) = F_Y(x) \quad \forall\, x \in \mathbb{R}$, i.e., MGF uniquely characterizes the distb^n.

- Two rvs X and Y are independent. Then,

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) \quad \forall\, t \in \mathbb{R}.$$

Characteristic func^n

$$\phi_X(\omega) = E\left[e^{j\omega X}\right], \quad \omega \in \mathbb{R}.$$

Properties:

(i) $\phi_X(0) = 1$

(ii) $|\phi_X(\omega)| \leq 1 \quad \forall\, \omega \in \mathbb{R}.$

(iii) If EX^n exists, then it can be obtained by

$$EX^n = (-j)^n \cdot \frac{d^n}{d\omega^n}\left(\phi_X(\omega)\right)\Big|_{\omega=0}.$$

(iv)

$$X \perp\!\!\!\perp Y \Rightarrow \phi_{X+Y}(\omega) = \phi_X(\omega)\, \phi_Y(\omega).$$

Note:

$X \perp\!\!\!\perp Y \iff \forall\, t_1, t_2 \in \mathbb{R}$,

$m_{\underline{X}}(\underline{t}) = m_X(t_1)\, m_Y(t_2)$, where

$$\underline{X} = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \underline{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \in \mathbb{R}^2.$$

$X \perp\!\!\!\perp Y \iff \forall\, \omega_1, \omega_2 \in \mathbb{R}$,

$\phi_{\underline{X}}(\underline{\omega}) = \phi_X(\omega_1)\, \phi_Y(\omega_2)$, where

$$\underline{X} = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \underline{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}.$$

- If $X \perp\!\!\!\perp Y$, then density of $X+Y$ = Convolution of f_X and f_Y. Converse not true. (HW4, Q1)

- Joint MGF and Char. func^n:

$\underline{X} = (X_1, \ldots, X_n)$,	$\underline{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$,	$\underline{\omega} = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$

$$m_{\underline{X}}(\underline{t}) = E\left[e^{\underline{t}^T \underline{X}}\right]$$

$$\phi_{\underline{X}}(\underline{\omega}) = E\left[e^{j\,\underline{\omega}^T \underline{X}}\right].$$

E2 202 Quiz 5 Nov 22, 2017

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1. Let $(N(t) : t \geqslant 0)$ be a homogeneous Poisson process with uniform rate λ. Let S_1 denote the first jump time. Given $t > 0$, find the following conditional distribution for all $s > 0$ and $n \in \mathbb{N}$,

$$P(S_1 > s | N(t) = n).$$

Characteristic function & joint characteristic func^n always exist

10. Jointly Gaussian rvs :

$X_1, \dots, X_n$ jointly Gaussian iff

$$\Phi_{\underline{X}}(\underline{\omega}) = e^{j\underline{\omega}^T \underline{\mu} - \underline{\omega}^T K \underline{\omega}/2}, \text{ where } \underline{\mu} = E\underline{X}.$$

$$K = E[(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})$$

Alternatively, $X_1, \dots, X_n$ jointly Gaussian iff every linear combination of $X_1, \dots, X_n$ is Gaussian, i.e., $a_1 X_1 + \dots + a_n X_n$ Gaussian $\forall a_1, \dots, a_n \in \mathbb{R}$.

Properties of K:

(i) K is symmetric. Thus, all its eigenvalues are real. Also, K is diagonalizable, i.e; $\exists$ an orthogonal matrix Q s.t.

$$QKQ^T = \text{diag}(\lambda_1, \dots, \lambda_n) \; ;$$

$\lambda_1, \dots, \lambda_n$ - eigenvalues

diag - matrix with only diagonal entries

(ii) K is +ve semidefinite, i.e;

$$a^T K a \geq 0 \quad \forall \underline{a} \in \mathbb{R}^n.$$

If K is +ve definite, then it is invertible, and a joint pdf for $X_1, \dots, X_n$ jointly Gaussian can be written as

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(\sqrt{2\pi})^n \sqrt{\det(K)}} e^{-\frac{(\underline{x}-\underline{\mu})^T K^{-1} (\underline{x}-\underline{\mu})}{2}} \quad \forall \underline{x} \in \mathbb{R}$$

• $X_1, \dots, X_n$ jointly Gaussian. Then

$X_1, \dots, X_n$ indep $\iff$ cov. matrix is diagonal.

Corollary: If X and Y are jointly Gaussian, then

$X \perp\!\!\!\perp Y \iff X$ and Y uncorrelated.

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11. Transformations:

$$T : (X_1, \dots, X_n) \longrightarrow (Y_1, \dots, Y_n)$$

$$f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) = \underbrace{\left| \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & & \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \right|}_{|\det(\text{Jacobian})|} \cdot f_{X_1, \dots X_n}(x_1, \dots, x_n)$$

---

12. Conditional expectations

$(\Omega, \mathcal{F}, P)$ given, $X : \Omega \to \mathbb{R}$ rv given.

Suppose $\mathcal{G} \subseteq \mathcal{F}$ is a sub σ-algebra of $\mathcal{F}$. Then,

$E[X|\mathcal{G}]$ is any rv Y which satisfies

(i) $Y \in \mathcal{G}$, i.e., Y is $\mathcal{G}$-measurable

(ii) $\forall A \in \mathcal{G}$,

$$\int_A X \, dP = \int_A Y \, dP$$

or $E[X \mathbb{1}_A] = E[Y \mathbb{1}_A]$.

Egs: • If $X \perp\!\!\!\perp \mathcal{G}$, then $E[X|\mathcal{G}] = E[X]$

• If $X \in \mathcal{G}$, then $E[X|\mathcal{G}] = X$.

• $E[X|Y] = E[X|\sigma(Y)]$, where $\sigma(Y)$ is the smallest σ-algebra wrt which Y is measurable, i.e.,

$$\sigma(Y) = \{A \in \mathcal{F} : \exists B \in \mathcal{B}(\mathbb{R}) \text{ s.t. } A = Y^{-1}(B)\}$$

See HW1, Q8.

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$$P(S_1 > s | N(t) = n).$$

---

12. Convergence

Four forms:

Almost sure: $P(X_n \to X) = 1.$

In prob: $P(|X_n - X| > \varepsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon > 0$

mean-squared: $E[(X_n - X)^2] \xrightarrow[n\to\infty]{} 0.$

In distribution: $F_{X_n}(x) \xrightarrow[n\to\infty]{} F_X(x) \quad \forall x \in C_{F_X}$

C_{F_X}: set of Continuity points of F_X.

Implications:

$$\begin{tikzcd} \text{a.s} \arrow[dr, Rightarrow] & & \\ & \text{i.p} \arrow[r, Rightarrow] \arrow[ul, bend right, "2"'] \arrow[dl, bend left, "3"] & d \arrow[l, Rightarrow, bend left, "1"] \\ \text{m.s} \arrow[ur, Rightarrow] & & \end{tikzcd}$$

① This is true only when $X_n \xrightarrow{d} c$, $c \in \mathbb{R}$.

② Borel-Cantelli Lemma helps us go from i.p. to a.s.

③ If $P(|X_n| \le L) = 1 \;\forall n \geq 1$, then

$$X_n \xrightarrow{i.p} X \Rightarrow X_n \xrightarrow{m.s.} X$$

Borel-Cantelli Lemma:

- Infinitely often set: Let $A_1, A_2, \ldots \in \mathcal{F}$. Then

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$ is called "A_n i.o." set.

or "ω i.o." set.

- All but finitely many set : $B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$.

Fact : $B \subseteq A$, i.e.,

NOTE : BCL is stated at the end of this doc.

$$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

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13. Exchanging limits and expectations $\left(\lim_{n\to\infty} E[X_n] = E\big[\underbrace{\lim_{n\to\infty} X_n}_{=X}\big] \right)$.

(i) MCT

(a) Suppose $X_n \geq Y$, $EY > -\infty$ and $X_n \overset{a.s.}{\uparrow} X$. Then $EX_n \uparrow EX$.

(b) Suppose $X_n \leq Y$, $EY < \infty$, and $X_n \underset{a.s.}{\downarrow} X$. Then, $EX_n \downarrow EX$.

Corollary : If $X_n \geq 0 \quad \forall\ n \geq 1$,

$$E\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} E[X_n] \quad \left(\text{used frequently for } X_n = \mathbb{1}_{A_n}\right).$$

(ii) DCT :

Suppose $|X_n| \leq Y$, $EY < \infty$ and $X_n \xrightarrow{a.s.} X$. Then,

(i) $E|X| < \infty$.

(ii) $EX_n \longrightarrow EX$.

---

14. Limit Theorems (WLLN, SLLN, CLT)

(i) WLLN : $X_1, \ldots, X_n$ iid , $E|X_1| < \infty$. Then

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{i.p.} E[X_1].$$

(ii) SLLN : $X_1, \ldots, X_n$ iid , $E|X_1| < \infty$: Then

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{a.s.} E[X_1].$$

(iii) CLT : $X_1, \ldots, X_n$ iid , $Var(X_1) < \infty$. Then

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow[n\to\infty]{d} N(0,1)$$

(Here, X_i's are assumed to have zero mean & unit variance wlog)

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$$P(S_1 > s | N(t) = n).$$

15. Random processes

- It is a collection of rvs $\{X_t : t \in T\}$. Here, T is an arbitrary index set.
  - Mean func^n $\mu_X(t) = E[X_t]$.
  - Correlation func^n $R_X(t, \tau) = E[X_t X_\tau]$.
- A random process is fully specified by its FDDs, i.e.,

$$F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}(x_1, \ldots, x_n) \quad \forall\ t_1, \ldots, t_n \in T,\ \forall\ x_1, \ldots, x_n \in \mathbb{R},\ \forall\ n \geqslant 1.$$

- A rp is said to be (strictly) stationary iff FDDs are timeshift invariant, i.e.,

$$F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}(x_1, \ldots, x_n) = F_{X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_n+\tau}}(x_1, \ldots, x_n)$$

$\forall\ x_1, \ldots, x_n \in \mathbb{R}$
$\forall\ t_1, \ldots, t_n \in T$
$\forall\ t_1+\tau, \ldots, t_n+\tau \in T$
$\forall\ n \geqslant 1$.

- A rp is said to be wide-sense stationary (wss) iff
  (i) μ_X is not a func^n of time
  (ii) $R_X(t, t+\tau)$ is a func^n only of τ $\forall\ t \in T,\ t+\tau \in T$

- Any iid process is stationary.

16. DTMCs and Bernoulli processes

- $\{X_n : n \geq 1\}$ iid Ber(p) process.
- N_n — # successes till time n.
- Success instants T_k.
time
- $\{T_k \leq n\} = \{N_n \geq k\}$; (T_k) and (N_n) are inverse processes.

Stopping times:

$(\Omega, \mathcal{F}, P)$ given.

. $(\mathcal{F}_t)_{t \in T}$ is a filtration - given

. A rv $\tau : \Omega \to T$ is called a stopping time wrt $(\mathcal{F}_t)$

if : $\{\tau \leq t\} \in \mathcal{F}_t \ \forall t \in T$.

Properties of stopping times:

- τ_1, τ_2 $\underbrace{\text{st}}_{\text{stop time}}$ $\Rightarrow$ $\underbrace{\tau_1 \vee \tau_2}_{\min\{\tau_1, \tau_2\}}$ is a stopping time
- τ_1, τ_2 st and T separable $\Rightarrow$ $\tau_1 + \tau_2$ stopping time

Stopping time σ-algebra

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \ \forall t \in T\}.$$

Wald's Lemma:

(i) Let (X_n) be iid, and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

If $P(N = n) = 1$, then

$$E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i]. = EN \cdot EX_1.$$

(ii) If $N \perp\!\!\!\perp X_1, X_2, \dots$ ~~then~~ and X_i are iid, then

$$E\left[\sum_{i=1}^{N} X_i\right] = EN \cdot EX_1$$

Wald: (iii) If N is a stopping time with $P(N < \infty) = 1$, then

$$E\left[\sum_{i=1}^{N} X_i\right] = EN \cdot EX_1.$$

E2 202 Quiz 5 Nov 22, 2017

Name: ______________________ SR No: ______________________

Question:	1	Total
Points:	5	5
Score:		

1. Let $(N(t) : t \geqslant 0)$ be a homogeneous Poisson process with uniform rate λ. Let S_1 denote the first jump time. Given $t > 0$, find the following conditional distribution for all $s > 0$ and $n \in \mathbb{N}$,

$$P(S_1 > s | N(t) = n).$$

Note: For Wald's lemma, N should be a stopping time wrt $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

DTMCs:

- $(X_n)_{n \geq 1}$ is a DTMC on a countable state space S iff
$$P(X_{n+1} = j \mid X_0 = i_0, \dots, X_n = i) = P(X_{n+1} = j \mid X_n = i)$$
$\forall\ i_0, i_1, \dots, i_{n-1}, j, i \in S$ and $\forall\ n \geq 1$.

- Time-homogeneity:
$$P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i) = p_{ij}$$

- TPM: An $|S| \times |S|$ matrix P whose $(i,j)^{th}$ entry is $(P)_{ij} = p_{ij}$.
Row sums of P are 1.

- Chapman-Kolmogorov equation:
$$p_{ij}^{(n)} = (P^n)_{ij}.$$

- Strong Markov property:
$$P(X_{N+k} = j \mid X_0 = i_0, \dots, X_N = i) = p_{ij}^{(k)}.$$

Hitting times & Recurrence times

$$H_j = \inf\{n \geq 1 : X_n = j\} \rightarrow \text{first hitting time for state } j.$$

$$f_{ij}^{(n)} = P(H_j = n \mid X_0 = i)$$

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}.$$

- If $f_{jj} < 1$, then j is called transient.

  $f_{jj} = 1$, j is called recurrent.

$$\mu_{jj} = \sum_{n=1}^{\infty} n.\, f_{jj}^{(n)}.$$

$\mu_{jj} < \infty$ +ve recurrent

$\mu_{jj} = +\infty$ Null recurrent

- j transient $\Longleftrightarrow \sum_{n=1}^{\infty} p_{jj} < \infty$ (BCL; see Mid term: Q1)

  j recurrent $\Longleftrightarrow \sum_{n=1}^{\infty} p_{jj} = +\infty$ (cannot use BCL since (X_n) is not indep seq, it is Markov!)

Communicating classes & class properties

- $i \rightsquigarrow j$ if $p_{ij}^{(n)} > 0$ for some $n > 0$.

- $i \leftrightsquigarrow j$ iff $i \rightsquigarrow j$ and $j \rightsquigarrow i$.

  $\leftrightsquigarrow$ partitions S into equivalence classes called Comm. classes.

- DTMC is said to be irreducible if only one comm class exists.

- Period of a state:

$$d(j) = \gcd\left(\{n \geq 1 : f_{jj}^{(n)} > 0\}\right).$$
$$= \gcd\left(\{n \geq 1 : p_{jj}^{(n)} > 0\}\right).$$

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Name: ______________________ SR No: ______________________

Question:	1	Total
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1. Let $(N(t) : t \geqslant 0)$ be a homogeneous Poisson process with uniform rate λ. Let S_1 denote the first jump time. Given $t > 0$, find the following conditional distribution for all $s > 0$ and $n \in \mathbb{N}$,

$$P(S_1 > s | N(t) = n).$$

- If $d(j) = 1$, j is called aperiodic state.

- Transience, recurrence & periodicity are class properties.

- ~~Open~~ A comm. class C is called ~~open~~ closed if there are no edges leaving C. A class which is not closed is said to be open.

- Open comm classes are transient
  Finite closed comm classes are +ve recurrent.

Invariant distributions

A prob distb^n $\underline{\pi}$ is said to be a stationary distb^n iff it satisfies $\underline{\pi} = \underline{\pi} P$.

- $(X_n)_{n\geq 1}$ DTMC irreducible. Then

  $(X_n)_{n \geq 1}$ +ve recurrent $\iff$ $\exists$ unique prob vector $\underline{\pi}$ s.t. $\underline{\pi} = \underline{\pi} P$.

- If $(X_n)_{n\geq}$ is aperiodic, irreducible & +ve recurrent, then

  $p_{ij}^{(n)} \longrightarrow \pi_j$ as $n \to \infty$ $\forall j \in S$. $i \in S$.

Consequently,

$\pi_j^{(n)} \longrightarrow \pi_j$ $\forall j \in S$.

distb^n of

- ~~One could use~~

Borel - Cantelli Lemma

$A_1, A_2, \ldots \in \mathcal{F}$

(i) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \text{ i.o.}) = 0$.

(ii) If A_n's are ~~pairwise~~ mutually independent ~~(i.e., $A_i \cap A_j = \phi \ \forall i \neq j$)~~ and $\sum_{n=1}^{\infty} P(A_n) = +\infty$, Then $P(A_n \text{ i.o.}) = 1$

Note: We don't require independence above. Actually, only pairwise independence suffices.